NATIONAL OPEN UNIVERSITY OF NIGERIA
University Village, Plot 91, Cadastral Zone, Nnamdi Azikwe Express Way, Jabi-Abuja
FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS
2024_2 EXAMINATION

Course Code: MTH412 **Course Title: FUNCTIONAL ANALYSIS II**
Credit Unit: 3 **Time Allowed: 3 Hours**
Total: 70 Marks
Instruction: Answer Question One (1) and Any Other 3 Questions

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1.(a) Given that $\{T_n\}$ is a sequence of bounded linear operator in $B(X,Y)$. When is T_n said to converge strongly to $T \in B(X,Y)$? **(4marks)**

(b) If $X^* \in R^n$ and $r > 0$, prove that the ball $B(X^*, r) = \{y \in R^n : k\, y - x * k < r\}$ centered at X^* of radius r is a convex set. **(11marks)**

(c) Let $1 \le p \le \infty$, and p' the exponent dual of p . Then, for all $x, y \in K^N$, prove that $\sum_{i=1}^{N} |x_i|\,|y_i| \le |x|_p |y|_{p'}$. **(7marks)**

2.(a) Let $M: X \to Y$. Explain an isometry of normed spaces X and Y, using its definition, translations and linear map. **(5marks)**

(b) Let X and Y be normed linear spaces and let $T: X \to Y$ be a linear map. Prove that the following are equivalent:

i. T is continuous at the origin 0 (in that sense that if $\{x_n\}$ is a sequence in X such that $x_n \to 0$ as $n \to 0$, then $TX_n \to 0$ in Y as $n \to \infty$);

ii. T is Lipschitz. i.e. there exists a constant $K \ge 0$ such that, for each $x \in X$, $\|Tx\| \le K\|x\|$. **(11marks)**

3.(a) Given that x and v are vectors in R^n. When is the line L through x said to be a convex set? **(3marks)**

(b) State and Prove Minkowski's Inequality. **(13marks)**

4.(a) When is the family of bounded linear map $B(X,Y)$ said to be a vector space? **(6marks)**

(b) Given that E is a vector space of K, when is a function $(.|.): E \times E \to K$ said to be an inner product space and what makes it a Hilbert space? **(10marks)**

5(a) Given that(X, ρ) is a metric space, when is open cover compactness defined on X? **(5marks)**

(b) Prove that the set $M := \{(2\pi)^{-1/2} e^{inx} : n \in Z\}$ forms an orthonormal system in $L_2((-\pi,\pi), C)$. **(11marks)**