**FACULTY OF SCIENCES**
**DEPARTMENT OF MATHEMATICS**
**2023_2 EXAMINATIONS_**

**Course Code:** MTH 312
**Course Title:** Group and Rings
**Credit Unit:** 3
**Time Allowed:** 3 Hours
**Total:** 70 Marks
**Instruction:** Answer Question Number One and Any Other Four Questions

1. Let $G$ be a group An isomorphism $\varphi: G \to G$ is called an automophism of $G$ and the set of all automorphisms of $G$ is denoted by $Aut(G)$.
   (a) Show that $Aut(G)$ is a group under composition of functions **(4 marks)**
   (b) Let $a \in G$ be fixed. Define the map $\theta_a: G \to G$ by $\theta_a(x) = ax^{a-1}$ for all $x \in G$. Show that $\theta_a \in Aut(G)$. **(4 marks)**
   (c) Let $Inn(G)$ be the set of all inner automorphisms of $G$. Show that $Inn(G)$ is a normal subgroup of $Aut(G)$. **( 4 marks)**
   (d) For $x, y \in G$, we say that $y$ is conjugate to $x$, if $\exists a \in G$ such that $y = \theta_a(x)$.
      i. Show that $\sim$ is an equivalence relation on $G$ **(4 marks)**
      ii. Let $H < G$. Show that set $\theta_a(x) | x \in H$ is a subgroup of $G$. **(3 marks)**
   (e) Show that $(nZ, +, .)$ is a ring where $n \in Z$. **(3 marks)**

2. (a) Obtain the subgroup of $S_4$, to which $Z_4$ is isomorphic. Is $Z_4 = A_4$ ? **(6 marks)**
   (b) Let $f: X \to Y$ and $g: Y \to Z$ between mappings such that $g \circ f$ is injective. Show that $f$ must be injective but $g$ need not be injective. **(6 marks)**

3. (a) Find the reminder when $a(x) = x^6 + 3x^6 - 3x + 2$ is divided by $b(x) = 3x^2 + 2x - 3$ in $Z_7(x)$. **(6 marks)**
   (b) Find the $g.c.d$ of $a(x) = x^6 + 3x^6 - 3x + 2$ and $b(x) = 3x^2 + 2x - 3$ in $Z_7(x)$ **(6 marks)**

4. (a) Find the polynomials $r(x)$ and $s(x)$ such that $d(x) = a(x) \cdot r(x) + b(x) \cdot s(x)$ **(6 marks)**
   (b) Show that the functions $f(x) = x^2$ and $g(x) = x^3$ commute with each other under composition. **(6 marks)**

5. (a) Show that a group $G$ order 255 $(3 \times 5 \times 17)$ has either 1 or 51 Sylow 5 subgroups **(6 marks)**
   (b) How many Sylow 3-subgroups can it have? **(6 marks)**

6. Consider the set $M_2(R) = \left\{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}\right\}$. Show that $M_2(R)$ is a ring with respect to addition and multiplication of matrices. **(12 marks)**