NATIONAL OPEN UNIVERSITY OF NIGERIA
Plot 91, Cadastral Zone, Nnamdi Azikwe Expressway. Jabi, Abuja.
FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS
2023_1 POP EXAMINATION

Course Code: MTH 412
Course Title: Functional Analysis II
Credit Unit: 3
Time Allowed: 3 Hours
Instruction: Answer Question number One and Any Other three Questions

1. (a) Let X be a linear space over a scale field $K = \mathbb{R}$ *or* $\mathbb{C}$, then define the following terms on the linear space:
   (i) a norm on a linear space over a scalar field **(5 marks)**
   (ii) l_∞ - space. **(4 marks)**

   (b) Let X be a linear space defined on a norm $k \cdot k$ such that $\rho: X \times X \to \mathbb{R}$ is defined for arbitrary $x, y \in X$ by $\rho(x, y) = kx - yk$. Prove that that ρ is a metric on X and also (X, ρ) is a metric space **(8 marks)**

   (c) Let matrix A represents G in the usual basis of C^3. Find the adjoint of $G: C^3 \to C^3$ defined by:

   (i) $G(x, y, z) = [3x + (1 - i)y, (2 + i)x - 3iz, 4ix + (3 - 2i)y - 3z]$; **(4 marks)**

   (ii) $F(x, y, z) = [-2x + (3 + i)y, (4 - i)x + 2iz, 5ix + (1 - 2i)y - 7z]$ **(4 marks)**

2. (a) Vectors x and v in $\mathbb{R}^n$, the line L along x in the direction of v is given as L = {x + av: , $a \in \mathbb{R}$}. Prove that L is convex. **(7 marks)**

   (b) Let $x = (1, 4, -5, -3)$ and $y = (-2, 1, 2, 3)$ in R^4. Evaluate
   (i) $kxk2$ **(4 marks)**
   (ii) $kyk2$ **(4 marks)**

3. (a) Prove that Bessel's inequality is orthogonal **(6 marks)**

(b) Let (X, ρ) be a complete metric such that $E \subset X$. Prove that (E, ρ_E) is complete if and only if it is closed. **(9 marks)**

4. (a) Let set A= {u_1, u_2, . . ., u_n} be an orthogonal vector. Prove that

   ku_1 + u_2 + . . . +u_r**k^2** = **k**u_1**k^2 + k**u_2**k^2** + . . . + **k**u_r**k^2.** **(9 marks)**

   (b) Mention 3 examples of incomplete normed linear spaces. **(6 marks)**

5. (a) Define Riesz Representation theorem on a Hilbert space. **(3 marks)**

   (b) Explain why a linear space is not a Hilbert space with its norm. **(5 marks)**

   (c) Let $u_1 = (2,4,2), u_2 = (4,2,-8)$ and $u_3 = (6,-4,2)$ are vectors in R^3. Show that the vectors are orthogonal **(7 marks)**

6. Let vector u^* in U such that k$x - u^*$k = $\inf_{u \in U} kx - uk$. Show that $u^* \in U$ is a unique minimizing vector if and only if $(x - u^*) \perp U$. **(15 marks)**