NATIONAL OPEN UNIVERSITY OF NIGERIA
University Village, Plot 91, Cadastral Zone, Nnamdi Azikiwe Expressway, Jabi, Abuja

FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS
2022_2 Examinations

Course Code:	**MTH301**
Course Title:	**Functional Analysis I**
Credit Unit:	**3**
Time Allowed:	**3 Hours**
Total:	**70 Marks**
Instruction:	**Answer Question One (1) and Any Other 3 Questions**

(1) (a) Define the followings:
(i) convergent Sequence in a metric space **(3 Marks)**
(ii) closed, open, infinite intervals on the real line. **(3 Marks)**

(b) Let (X,d) be a metric space. Prove that a subset A of X is closed in (X, d) if and only if every convergent sequence of points in A, converges to a point in A. In particular, prove that A is closed in (X, d) if and only if $a_{n \to x,}$ where $x \in X$ and a_n is a sequence of points in A, for all n, implies that $x \in A$. **(9 Marks)**

(c) Let (X, d) and (Y, d_1) be metric spaces and f a mapping of X into Y. Let τ and τ_1 be the topologies determined by d and d_1 respectively. Prove that $f: (x, \tau) \to (y, \tau_1)$ is continuous if and only if $x_n \to x \to f(x_n) \to f(x)$. That is if $x_1, x_2, \ldots x_n \ldots$ is a sequence of points in (X, d) converging to x, then the sequence of points $f(x_1), f(x_2), \ldots f(x_n) \ldots in\ (Y, d_1)$ converges to x. **(10 Marks)**

(2) (a) Define the followings:

(i) a metricable topological space **(3 Marks)**
(ii) Euclidean space **(3 Marks)**
(iii) Absolute value **(3 Marks)**
(iv) Norm of a vector. **(3 Marks)**

(b) Let $x \in \Re^n$ and $\mathcal{E} > 0$. Prove that the set $B(x, \varepsilon) = \{y \in \Re^n : d(x, y) < E\}$ is open. **(3 Marks)**

(3) Define the following
(i) an interior point **(1 marks)**
(ii) boundary point **(2 marks)**
(iii) closure of a subset of a set **(3 marks)**
(iv) accumulation point **(3 marks)**
(v) closed set **(3 marks)**
(vi) interior of a set. **(3 Marks)**

(4) (a)(i)Totally bounded metric space **(3 Marks)**

(ii) Sequentially compact metric space **(3 Marks)**

(b) Let X be a metric space and let Y be a subspace of X. Prove that

(i) If X is compact and Y is closed in X, then Y is compact. **(4 Marks)**

(ii) If Y is compact, then it is closed in X. **(5 Marks)**

(5) a(i) Define a topological space **(4 Marks)**

(ii) Give three examples of topological spaces **(6 Marks)**

(b) Let (K, d_K) be a compact metric space. Let (Y, d_Y) be any metric space and let $f: K \to Y$ be continuous. Prove that f(K) is compact. **(5 Marks)**

(6) a(i) If (A, d_A) *and* $(B,\ d_B)$ are metric spaces, when is a function

$f: A \to B$ said to be continuous? **(3 Marks)**

(ii) Let f and g be real-valued functions with domain(f) = Range (g) = D $\subset \mathcal{R}^N$. If $\lim_{x \to x_0} f(x) = l$ *and* $\lim_{x \to x_0} g(x) = m$ then state the three limit theorems concerning the sum, product and quotient of the above function.
(6 Marks)

(b) Given the function $f: \mathfrak{R}^2 \to \mathfrak{R}$ *and* $(x_0, y_0) = (1,3)$. Compute the limits of the following functions as $(x, y) \to (1,3)$:

(i) $\lim_{(x_0,y_0)\to(1,3)} f(x,y), if\ f(x,y) = \frac{2x}{x^2+y^{2+1}}$. **(3 Marks)**

(ii) $\lim_{(x_0,y_0)\to(1,3)} f(x,y), if\ f(x,y) = x^2 + y^2 + 1.$ **(3 Marks)**