NATIONAL OPEN UNIVERSITY OF NIGERIA
University Village, Plot 91, Cadastral Zone, Nnamdi Azikwe Expressway. Jabi, Abuja.

FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS
2021_2 Examinations

Course Code: **MTH 412**
Course Title: **Functional Analysis II**
Credit Unit: **3**
Time Allowed: **3 Hours**
Instruction: **Attempt Number One (1) and any other Four (4) Questions**

1. (a) Find the adjoint of the following functions $G, F: C^3 \to C^3$ defined by:
(i) G(x, y, z) = [x + (1 – 2i)y, (5 + i)x – 4iz, 7ix + (3 – 3i)y + 2z] **(4 marks)**
(ii) F(x, y, z) = [-4x + (9 - i)y, (1 - i)x - 3iz, 2ix + (1 + 3i)y – 8z] **(4 marks)**
(b) State Hahn – Banach Theorem. **(3 marks)**
(c) Let T be an operator on a Hilbert space H. Show that the following are equivalent:
(i) $T^*T = I$ **(2 marks)**
(ii) hTx, Ty**i** = hx, y**i** **(2 marks)**
(iii) KTx K = KxK for all x $\in H$. **(2 marks)**
(d) Let S consists of the following two
vectors in $\mathbb{R}^2$:
$u_1 = (9, 3)$, $u_2 = (4, -12)$.
Verify that the vectors are orthogonal and hence they are linearly independent. **(5 marks)**
2. (a) State polarization identity. **(5 marks)**
(b) Show that the collection of self – adjoint operators on H forms a closed, real linear subspace of B(H). **(7 marks)**
3. (a) Consider the basis $\{v_1 = (-1, 2), v_2 = (1, 5)\}$ of R^2. Find the dual basis $\{f_1, f_2\}$ of (R^2) such that $f_1(v_1) = 1$, $f_1(v_2) = 0$, $f_2(v_1) = 0$ $f_2(v_2) = 1$ by finding the linear functional $f_1(x, y) = ax + by$ and $f_2(x, y) = cx + dy$. **(10 marks)**
(b) Let U and V be arbitrary subspaces of a Hilbert space H. Show that
$U \subset U^{\perp\perp}$; **(2 marks)**
4. (a) What is inner product space? **(7 marks)**
(b) State Bessel's inequality. **(5 marks)**

5. (a) Define the following terms: (i) normal operator **(2 marks)**
(ii) Equivalent norm **(3 marks)**
(b) Let k . k be a norm defined on a linear space X. If $\rho: X \times X \to \mathbb{R}$ is defined for arbitrary $x, y \in X$ by
$\rho(x, y)$ = kx – yk.
Show that ρ is a metric on X and so (X, ρ) is a metric space. **(7 marks)**

6. Let X and Y be two linear spaces over scalar field, K, and let $T : X \rightarrow Y$ be a linear map. Show that

(i) $T(0) = 0$; **(2 marks)**

(ii) The range of T , $R(T) = \{y \in Y : Tx = y$ for some $x \in X$ is a linear subspace of Y ; **(5 marks)**

(iii) T is one-to-one if and only if $T(x) = 0$ implies $x = 0$. **(5 marks)**