**NATIONAL OPEN UNIVERSITY OF NIGERIA**
**FACULTY OF SCIENCES**
**DEPARTMENT OF MATHEMATICS**
**2020_1 EXAMINATION**

**Course Code: MTH 412**
**Course Title: Functional Analysis**
**Credit Units: 3**
**Time Allowed: 3 Hours**
**Instruction: Answer Question Number One (1) and Any Other Four (4) Questions**

1. (a) Define (i) Normed linear space **(4 marks)**
(ii) Equivalent norms in a linear space **(2 marks)**

(b) (i) Let $\|.\|$ be a norm defined on a linear space $X$ . If $d: X \times X \rightarrow \Re$ is defined for arbitrary $x, y \in X$ by $d(x,y) = \|x-y\|$ prove that $d$ is a metric on $X$ and as such $(X,d)$ is a metric space. **(5 marks)**

(ii) Verify that the real line $\Re$ becomes a normed linear space if we set $\|x\| = |x|$ for every number $x \in \Re$ **(4 marks)**

(c)(i) Prove that all norms defined on a finite dimensional space are equivalent. **(7 marks)**

2. (a) Define the following concepts:
(i) convex subspace of a linear space. **(2 marks)**
(ii) Line segment joining two points in a linear space **(2 marks)**
(b)Let $X$ and $V$ be two vector spaces in $\Re^n$. Prove that the line L through $X$ In the direction of V given by $L = \{x + \alpha V : \alpha \in \Re\}$ is a convex set. **(4 marks)**
(c)Prove that a nonempty subset C of a vector space X is convex if and only if C contains all convex combinations of all its points. **(4 marks)**

3. (a) Define the following concepts:
(i) Convergence sequence of a metric space **(2 marks)**
(ii) Cauchy sequence in a metric space **(2 marks)**
(iii) Banach space. **(2 marks)**

(iv) Give an example of incomplete normed linear space. **(2 marks)**

(b) Prove that the space $C[a,b]$ of continuous real valued functions defined $[a,b]$ is complete if it is endowed with the supremum norm
$\|.\|_0 = \max_{a \le t \le b} |f(t)|$ . **(4 marks)**

4. (a) Define the following

(i) Linear map **(2 marks)**

(ii) Linear functional **(2 marks)**

(iii) Bounded maps **(2 marks)**

(b) Let $X$ *and* $Y$ be two normed linear spaces and let $T : X \to Y$ be a linear map. Prove that the following are equivalent: (1) T is continuous (2) T is continuous at the origin (3) T is Lipschitz (4) If $D := \{x \in X : \|x\| \leq 1\}$ is closed unit disc in X, then T(D) is bounded, that is, there exists a constant $M \geq 0$ such that $\|T_x\| \leq M$.

**(6 marks)**

5. (a) (i) Define the Topological dual of a normed linear space **(2 marks)**

(ii) State the Hahn – Banach theorem **(2 marks)**

(iii) Explain uniform convergence of bounded linear operators. **(2 marks)**

(b) Let X be a normed linear space and F be its scalar field. Suppose that $\{x_1, x_2, \ldots, x_n\}$ is a basis of $X$ *over* $F$. Let $f_1, f_2, \ldots, f_n \in X$ be the linear functionals defined by

$$f_i(x_j) = \begin{cases} 1, & if \;\; i = j \\ 0, & if \;\; i \neq j \end{cases}$$

Prove that $\{f_1, f_2, \ldots, f_n\}$ is a basis for X. **(6 marks)**

6. (a) Define (i)The graph of a linear operator in normed linear spaces

**(3 marks)**

(ii) Closed linear operator **(3 marks)**

(b) Let T be the differentiation operator. Prove that

(i) T is linear **(2 marks)**

(ii) T is closed **(2 marks)**

(iii) T is not bounded. **(2 marks)**