NATIONAL OPEN UNIVERSITY OF NIGERIA
University Village, Plot 91, Cadastral Zone, Nnamdi Azikwe Expressway, Jabi, Abuja

FACULTY OF SCIENCES
DEPARTMENT OFMATHEMATICS

Course Code: **MTH381**
Course Title: **Mathematical Methods III**
Credit Unit: **3**
Time Allowed: **3 Hours**
Total: **70 Marks**
Instruction: **Answer Question** Number One **and any Other Four Questions**

1(a) (i) write the quotient $\frac{1+i}{\sqrt{3}-i}$ in polar form. **(2 marks)**

(ii) compute $\frac{1+i}{\sqrt{3}-i}$ **(1 marks)**

(iii) Find all the cube roots of $\sqrt{2}+i\sqrt{2}$ **(2 marks)**

1 (b) (i) Evalaute $|\bar{z}^2-2-(-2+2i)|$ **(2 marks)**

(ii) Evalaute $\left||\bar{z}^2-2)|-\sqrt{8}\right|$ **(2 marks)**

(iii) Let $f(z)=\bar{z}e^{-|z|^2}$. Determine the points at which $f'(z)$ exists, and find $f'(z)$ at these points. **(2 marks)**

(iv) Find an analytic function f whose imaginary part is given by $e^{-y}sinx$. **(2 marks)**

1 (c) (i) Use Demoivre's theorem with $n=4$ to prove that $cos4\theta=8cos^4\theta-8cos^2\theta+1$

And deduce that $cos\frac{\pi}{8}=\left(\frac{2+2\sqrt{2}}{4}\right)^{\frac{1}{2}}$. **(2 marks)**

(ii) Find an analytic function of $z=x+iy$ whose imaginary part is **(2 marks)**

$$(ycosy+xsiny)expx$$

(iii) Find the radius of convergence of the following Taylor series: **(1 marks)**

$$\sum_{n=1}^{\infty} z^n n^{\ln n}$$

1 (d) (i) Find the residue at each of the singularities of $f(z)=\frac{e^{z^3}}{z(z+1)}$. The function $f(z)$ has simple poles at $z=0$ and $z=-1$. Therefore, we have

$$R[f,0]=\lim_{z\to 0} zf(z)=\lim_{z\to 0}\frac{e^{z^3}}{z(z+1)}=1$$

And $R[f,-1]=\lim_{z\to -1}(z+1)f(z)=\lim_{z\to -1}\frac{e^{z^3}}{z}=-e^{-1}$ **(1 mark)**

(ii) Find the residue of $f(z)=\frac{cotz}{z^2}$ at $z=0$. **(1 marks)**

(iii) Evaluate $I=\int_0^{2\pi}\frac{1}{1+asin\theta}d\theta,\ 0<|a|<1$. **(2 marks)**

2 (a) (i) Evalaute $\int_0^1 \frac{1}{x^{\frac{1}{5}}} dx$ **(2 marks)**

(ii) show that $\int_{-\infty}^{\infty} \frac{x}{x^3 - a^3} dx = \frac{\pi}{\sqrt{3a}}; a > 0$ **(1 mark)**

(iii) show that $\int_{-\infty}^{\infty} \frac{\sqrt{x}}{x^3+1} dx = \frac{\pi}{3}$ **(2 marks)**

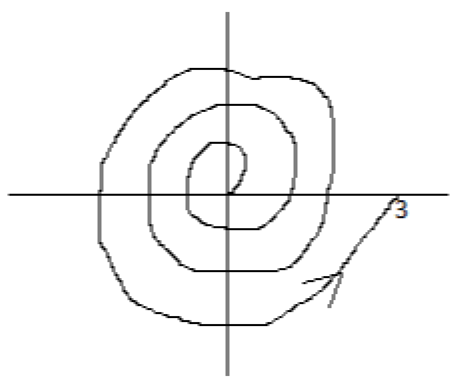
2 (b) (i) Evalaute $\int_\gamma \ f(z)dz$. If $f(z) = z - 1$ and γ is the curve given by

$z(t) = t + it^2, \quad 0 \le t \le 1.$ **(1 mark)**

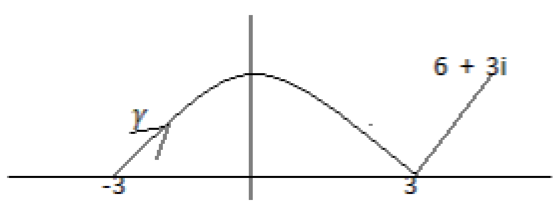
(ii) Let $f(z) = z - 1$ and $\gamma = \gamma_1 + \gamma_2$ where γ_1 is given by $z_1(t) = t,\ ,\ 0 \le t \le 1$ and γ_2 is given by $z_2(t) = 1 + i(t - 1),,\ 1 \le t \le 2$, then evaluate $\int_\gamma \ f(z)dz$ **(1 mark)**

(iii) let γ be given by $z(t) = 2e^{it},\ 0 \le t \le 2\pi$. Show that $\left|\int_\gamma \frac{e^z}{z^2+1} dz\right| \le \frac{4\pi e^2}{3}$ **(1 mark)**

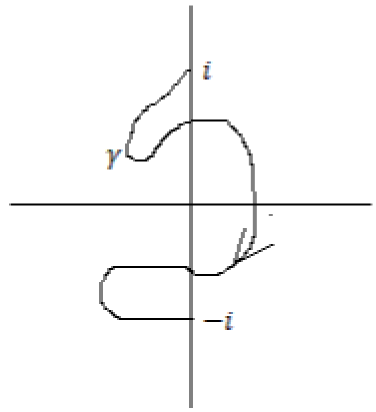
2 (c) (i) compute the integral $\int_\gamma \ (z^2 - 1)dz$, where γ is the contour in **(1 mark)**



(ii) compute the integral $\int_\gamma \ sinzdz$, where γ is the contour **(1 mark)**

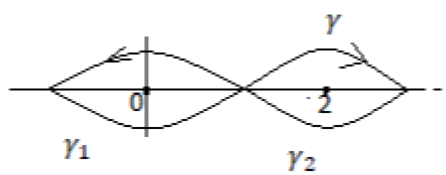


(iii) Compute $\int_\gamma \frac{dz}{z}$, where γ is the contour in the given figure **(2 marks)**

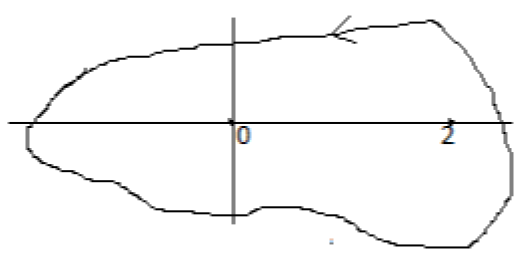


3. (a) Let $\gamma = \gamma_1 + \gamma_2$, where γ_1 and γ_2 respectively are given by $z_1(t) = ti$ and $z_2(t) = t + i$, $t \in [0,1]$. Furthermore, let $f(z) = (y - x) + 3ix^2$. Show that the function f cannot have an antiderivative. **(4 marks)**

3 (b) (i) Compute $\int_\gamma \frac{e^z}{z(z-2)} dz$, where γ is the following contour **(4 marks)**

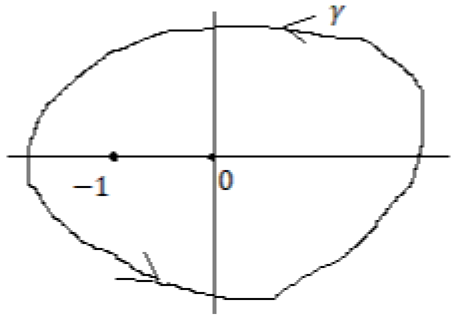


(ii) compute $\int_\gamma \frac{e^z}{z(z-2)} dz$, where γ is the following contour **(4 marks)**



4 (a) (i) Compute $\int_\gamma \frac{3z+1}{z(z-2)^2}dz$ along the contour γ given in 3b (i) above. **(2 marks)**

(ii) Compute $\int_\gamma \frac{\cosh z}{z(z-2)^2}dz$ where the contour γ is given in the figure below **(3 marks)**



(iii) Evaluate the function $f(z) = \int_0^1 e^{-z^2 t}dt.$ Let γ be any simple closed contour in the complex plane. Changing the order of integration, we have **(2 marks)**

4 (b) Find the Taylor expansion up to quadratic terms in $x-2$ and $y-3$, of $f(x,y) = ye^{xy}$ **(2 marks**

4 (c) Find and evaluate the maxima and saddle points of the function

$f(x,y) = xy(x^2 + y^2 - 1)$ **(3 marks)**

5 (a) (i) by finding $\frac{dI}{dy}$, evaluate the integral **(3 marks)**

$$\int_0^\infty \frac{e^{-xy}\sin x}{x}dx$$

(ii) show that the function $1, x, \sin x$ are linearly independent **(2 marks)**

(iii) Find the Laplace transform of the function $f(t) = e^{at}(\mathbf{2\ marks})$

5 (b) (i) Evaluate the double integral

$I = \iint_R x^2 y dxdy$ **(2 marks)**

where R is the triangular area bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$. Reverse the order of integration and demonstrate that the same results is obtained.

(ii) Find the Laplace transform of $\frac{d^2 f}{dt^2}$ **(3 marks)**

6 (a) In spherical polar coordinates $r, \theta, \emptyset$ the element of volume for a body that is symmetrical about the polar axis is $dV = 2\pi r^2 \sin\theta dr d\theta$, whilst its element of surface area is $2\pi r \sin\theta[(dr)^2 + r^2(d\theta)^2]^{\frac{1}{2}}$. A particular surface is defined by $r = 2a\cos\theta$, where a is a constant and $0 \leq \theta \leq \frac{\pi}{2}$. Find its total surface area and the volume it encloses, and hence identify the surface. **(7 marks)**

6 (b) By transforming to cylindrical polar coordinates, evaluate the integral

$$I = \iiint \ln(x^2 + y^2) dx dy dz$$

Over the interior of the conical region $x^2 + y^2 \leq z^2, 0 \leq z \leq 1$ **(5 marks)**