COURSE GUIDE

MTH 281
MATHEMATICAL METHODS I

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Printed 2013, 2017

ISBN: 978-058-069-1

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INTRODUCTION

The course is purposely for students of mathematics, physical sciences at undergraduate level.

It is assumed that the students have got enough mathematical background at 100 level and therefore fairly familiar with such topics as simple differentiation and integration, the use of trigometry identities, exponential and logarithmic functions.

The problems and worked examples in this course are purely mathematical to avoid the course being useful only to a section of scientists.

The course is a must for all students who will like to make career in mathematics and engineering.

COURSE AIMS

The course aims at giving you a good understanding of various methods in advanced mathematics.

This could be achieved through the following measures:

- Introducing you into limiting processes and continuity and differentiability.
- Introducing you to partial differentiation.
- Explaining the convergence of infinite series.
- Applying the knowledge in some special type of series such as Taylor and Maclaurin series.
- Cumulate the knowledge acquired in solving numerical some integration problems that cannot be solved analytically.

COURSE OBJECTIVES

At the end of this course, you should be able to:

- find limit define continuity and find derived functions of given mathematical functions
- be able to define convergence of infinite series and apply to some special series such as Taylor and Maclaurin series
- solve integration using material procedure and apply solve problems on mathematical methods correctly.

WORKING THROUGH THE COURSE

COURSE MATERIAL

STUDY UNITS

Unit 1 Limit, Continuity and Differentiability
Unit 2 Partial Differentiation
Unit 3 Convergence of Infinite Series
Unit 4 Taylor and Maclaurin Series
Unit 5 Numerical Integrations

While the first four units concentrate on mathematical methods and procedures the last units is on application of the method learn so far.

ASSESSMENT

The assessments of this course are therefore. There are graded exercises which are meant to and understanding as you progress in this course while the Tutor Marked Assignment are meant to be part of your final assessment.

The final assessment is at the end of the course assessment. It constitute 70% of the total grade for the course.

You are to read and master each unit carefully before progressing to other units.

MAIN COURSE

CONTENT **PAGES**

MODULE 1

UNIT 1 LIMITS, CONTINUITY AND DIFFERENTIABILITY

CONTENTS

1.0 INTRODUCTION

Recall that in MTH102, the idea of a limit was introduced. For example, it was shown that as θ becomes small $\frac{\sin\theta}{\theta}$ approaches unity. We will consider in this unit a more detailed and rigorous definition of the limit of a function. We will also study the concept of continuity and state sit of conditions when a function will be discontinuous. The two ideas of limit and continuity would be applied to establish a more rigorous definition of differentiability.

2.0 OBJECTIVES

After studying this unit, you should be able to:

- establish the limit of functions
- determine the continuity or otherwise of a function
- carry out the differentiation of a function
- apply the rolles and mean-value theorem to solutions of some problems
- be able to obtain n^{th} differentials coefficients of some simple functions by application of Leibnitz's formula.

3.0 MAIN CONTENT

3.1 Limits

Suppose f(x) is a given function of x. then, if we can make f(x) as near as we please to a given number l by choosing x sufficiently near to a number a, l is said to be the limit of f(x) as $x \to a$, and is written as

$$\lim_{x\to a} f(x) = l \tag{1}$$

It is important to emphasise the following points:

(a) the independent variable x may approach the point a either from left to right (that is, from $-\infty$ to a) or from right to left (from a to ∞). In many cases the limits of the function obtained in these two ways are different, and when this is the case we write them as

$$\lim_{x\to a-} f(x) = l_1, \qquad \lim_{x\to a+} f(x) = l_2,$$

respectively.

For example, the function $y = \tan^{-1}\frac{1}{x}$ tends to $\frac{\pi}{2}$ when x approaches zero from the positive side, and to $-\frac{\pi}{2}$ when x approaches zero from the negative side.

Consequently, we write

$$\lim_{x\to 0+} \tan^{-1}\frac{1}{x} = \frac{\pi}{2}, \quad \lim_{x\to 0-} \tan^{-1} = -\frac{\pi}{2}.$$

Sometimes we are faced with a function, which becomes arbitrarily large when x is chosen sufficiently close to a number a. when this happens we write

$$\lim_{x\to a} f(x) = \infty \qquad (2)$$

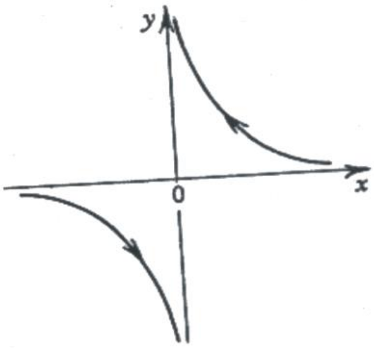


Fig.1.1

For example, the function $y = \frac{1}{x}$ tends to ∞ when $x \to 0$ from the positive side, and to $-\infty$ when $x \to$
from the negative side (see fig 1.1). Accordingly

$$\lim_{x\to 0+} \frac{1}{x} = \infty, \quad \lim_{x\to 0-} \frac{1}{x} = -\infty.$$

In all cases when the limits as $x \to a$ from both directions are equal (say l) we simply write

$$\lim_{x\to a} f(x) = l \qquad (3)$$

(b) in proceeding to the limit of f(x) as $x \to a$ we have to exclude x from becoming equal to *a* for two reasons. Firstly, the value of the function may not be defined at x = a, as, for example. $\frac{\sin x}{x}$ at x = 0. Secondly, if f(x) is defined at x = a its value may not be equal to $\lim_{x\to a} f(x)$. For example, if f(x) is defined by

$$f(x) = \begin{cases} 1 \text{ for } x \leqq 1, \\ \frac{1}{2} \text{ for } x > 1, \end{cases} \qquad (4)$$

(see Fig. 1.2) then

$$\lim_{x\to 1+} f(x) = \tfrac{1}{2}.$$

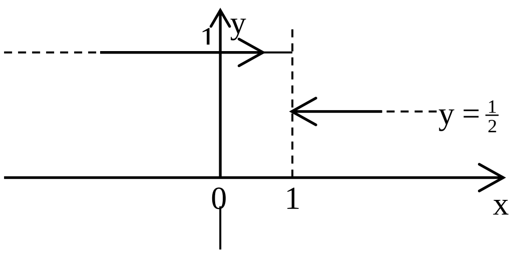


Fig. 1.2

This is not equal to the value of the function at x = 1, which by (4) is equal to unity.

The function y = cos (1/x) (see Fig. 1.3) is not only undefined

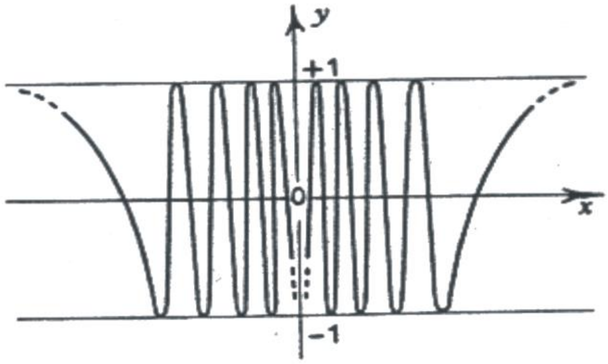


Fig 1.3

At x = 0, but possesses no limit there either, since as $x \to 0$ the graph oscillates infinitely many times between +1 and -1. the function therefore does not approach any particular value as $x \to 0$. However, $y = \cos\frac{1}{x}$ (Fig. 1.4)

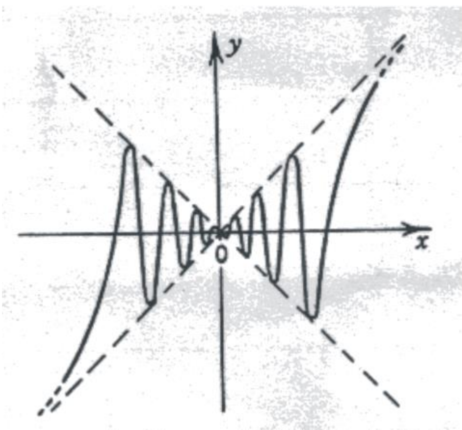


Fig 1.4

Although again oscillating infinitely many times as $x \to 0$ nevertheless does posses a limit in virtue of the factor x in front of the cosine term which decreases to zero in the limit. The limit of this function as x $\rightarrow$ 0 is therefore zero.

A more rigorous definition of the limit of a function is as follows: if f(x) tends to a limit l as x $\rightarrow$ a, then for any number ε (however small) it must be possible to find a number ε such that

In general the value of ε depends on the value of ε. Consider f(x) = 1 - $\frac{1}{x+2}$. According to (5) we are permitted to say that $\lim_{x\to1} f(x) = \frac{2}{3}$ provided a value of ε exists such that, for any ε,

$$\left|1-\frac{1}{x+2} - \frac{2}{3}\right| < \varepsilon,$$

when $$|x - 1| < \eta \qquad (6)$$

Suppose we take $\varepsilon = 10^{-3}$. Then

$$0.334 > \frac{1}{x+2} > 0.332 \text{ (to 3 decimals)},$$

which gives $0.994 < x < 1.010$.

For (6) to be satisfied we need therefore only take

$$(1.010 - 1) < \eta \qquad (7)$$

or $\eta > 0.010$. Hence, since the conditions (5) can be satisfied, the limit of f(x) as x $\rightarrow$ 1 exists and is equal to $\frac{2}{3}$.
We now state without proof three important theorems on limits. If f(x) and g(x) are two functions of x such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists, then

Theorem 1:

$$\lim_{x\to a} \{f(x) + \lim_{x\to a} g(x)\} = \lim_{x\to a} f(x) + \lim_{x\to a} g(x),$$

Theorem 2:

$$\lim_{x\to a} \{f(x)g(x)\} = \lim_{x\to a} f(x). \lim_{x\to a} g(x),$$

Theorem 3:

$$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} f(x)}$$

provided $\lim\limits_{x\to a} g(x) \neq 0$.

These theorems maybe readily extended to any finite number of functions.

The following example illustrates the use of these theorems.

Example 1: Suppose we wish to evaluate

$$\lim_{x\to 1}\left(\frac{x^m - 1}{x - 1}\right),$$

where m is a positive integer. Dividing the denominator into the numerator, the limit maybe written as

$$\lim_{x\to a}\ (1 + x + x^2 + \ldots\ldots + x^{m-1}), \qquad (8)$$

which by Theorem 1 is the same as

$$\lim_{x\to 1} 1 + \lim_{x\to 1} x + \lim_{x\to 1} x^2 + \ldots + \lim_{x\to 1} x^{m-1}. \qquad (9)$$

The value of each of these limits is unity and since there are m of them, the sum is m. Similarly, if m is a negative integer, say –k, where k is a positive integer, then

$$\lim_{x\to 1}\left(\frac{x^m - 1}{x - 1}\right) = \lim_{x\to 1}\left(\frac{x^{-k} - 1}{x - 1}\right) = \lim_{x\to 1}\left(\frac{1/x^k - 1}{x - 1}\right) = \lim_{x\to 1}\left(\frac{1 - x^k}{x - 1\ x^k}\right)$$

$$= -\lim_{x\to 1}\left\{\left(\frac{x^k - 1}{x - 1}\right)\frac{1}{x^k}\right\}. \qquad (10)$$

By Theorem 2, (10) may be written as

$$\lim_{x\to 1}\left(\frac{x^m - 1}{x - 1}\right) = -\lim_{x\to 1}\left(\frac{x^k - 1}{x - 1}\right)\ \lim_{x\to 1}\frac{1}{x^k} = -k = m, \qquad (11)$$

(making use of the result for a positive integer).
Likewise, if m is fractional, say p/q, where p and q are integers, then

$$\lim_{x\to 1}\left(\frac{x^m-1}{x-1}\right) = \lim_{x\to 1}\left(\frac{x^{p/q}-1}{x-1}\right) \qquad (12)$$

Now putting $x^{1/q} = y$ so that $x = y^q$ we have

$$\lim_{x\to 1}\left(\frac{x^m-1}{x-1}\right) = \lim_{y\to 1}\left(\frac{y^p-1}{y^q-1}\right) = \lim_{y\to 1}\left\{\frac{\left[\left(\frac{y^p-1}{y-1}\right)\right]}{\left[\left(\frac{y^q-1}{y-1}\right)\right]}\right\}. \qquad (13)$$

By Theorem 3, therefore

$$\lim_{x\to 1}\left(\frac{x^m-1}{x-1}\right) = \frac{\lim_{y\to 1}\left(\frac{y^p-1}{y-1}\right)}{\lim_{y\to 1}\left(\frac{y^q-1}{y-1}\right)} = \frac{p}{q} = m \qquad (14)$$

as before. Hence for all rational values of m

$$\lim_{x\to 1}\left(\frac{x^m-1}{x-1}\right) = m \qquad (15)$$

SELF ASSESSMENT EXERCISES 1

Evaluate the following limits:

a) $\lim_{x\to\infty}\frac{x+1}{x+2}$

b) $\lim_{x\to\infty}\frac{x^3+3}{2x^3+4x+1}$

c) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

d) $\lim_{x\to 0}\frac{\sec x-\cos x}{\sin x}$

e) $\frac{1+\cos\pi x}{\tan^2\pi x}$

3.2 Continuous and Discontinuous Functions

A single-valued function of x is said to be continuous at x – a if

(a) $\lim_{x \to a} f(x)$ exists,

(b) the function is defined for the value x = a, and

(c) if $\lim_{x \to a} f(x) = f(a)$.

When a function does not satisfy these conditions it is said to be discontinuous and x = a is called a point of discontinuity. In general, if the graph of a function has a break in it at a particular value of x it is discontinuous at that point. For example, the function y =1/x represented in Fig. 1.1 is discontinuous at x = 0, whilst the function defined by 1.1 (3) and represented in Fig. 1.2 is discontinuous at x = 1. There is, however, a slight difference between these two examples. The first function (y = 1/x) becomes infinite at the point of discontinuity and is said to have an infinite discontinuity at x = 0; the second function remains finite at the discontinuity and is therefore said to have a finite discontinuity at x = 1.

Functions like $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$ are discontinuous at x = 0 since they are not defined there (see condition (b) above).

It is an important result (and one that we shall need later on) that every polynomial of any degree is continuous for all x.

To prove this consider a polynomial of degree n

$$P_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n \qquad (16)$$

and take as a function f(x) any typical term $x^m (m \leqq n)$ in the polynomial. Then for nay arbitrary value of x, say x = a, $f(a) = a^m$. Now by theorem 2, (3.1)

$$\lim_{x \to a} f(x) = \lim_{x \to a} x^m = \left(\lim_{x \to a} x\right)^m = a^m = f(a). \qquad (17)$$

Hence the function x^m is continuous at x = a, and since a is arbitrary, it must be continuous for all x. This result applies to every term of the polynomial, and hence every polynomial is continuous for all x. An immediate consequence of this result is that every rational function (see

Chapter 1, 1.3 (e)) is continuous everywhere except at the points where the denominator vanishes. For example,

$$y = \frac{5x^2 + 3}{x-1 \quad x-2} \tag{18}$$

is continuous everywhere except at x = 1 and x = 2. The discontinuities are shown graphically by the existence of asymptotes at these values of x.

In general, the sums, differences, products and quotients of continuous functions (except, of course, at the zeros of the denominator in the case of a quotients).

SELF ASSESSMENT EXERCISES 2

Find the points of discontinuity of the following functions.

i) $\dfrac{x^3 + 4x + 6}{x^2 - 6x + 8}$

ii) sec x

iii) $\dfrac{\sin x}{\sqrt{x}}$

3.3 Differentiability

Consider a function y = f(x) whose graph is represented in fig. 3.5, and let P be a typical point on the curve with coordinates *(x,y)*. The coordinates of a neighbouring point Q can be written as ($x + \delta x$, $y + \delta y$), where the small change δx in x produces the small change δy in y. The expression

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \tan QPS \tag{19}$$

is then the slope of the straight line joining the points P and Q, and may be thought of as the mean value of the gradient of the curve y = f(x) in the range (x, $x + \delta x$). As the point Q approaches P, (19) may approach a limiting value given by

$$\lim_{\delta x \to 0}\left(\frac{f(x+\delta x) - f(x)}{\delta x}\right) = i \text{ (say)} \tag{20}$$

If this limit exists then geometrically this implies the existence of a tangent such that $l = \tan\theta$, where θ is the angle between the tangent at P and the x-axis. We refer to (20) as the differential coefficient of y with respect to x and denote it by the symbol $\frac{dy}{dx}$. Sometimes, however, it is convenient to denote $\frac{dy}{dx}$ by f'(x), $\frac{df}{dx}$, or by Dy or Df, where D is the operator $\frac{d}{dx}$.

A function y = f(x) is said to be differentiable if it possesses a differential coefficient, and to be differentiable at a point x = a if $\frac{dy}{dx}$ (or f'(x)) exists at that point.

From the definition of the differential coefficient as a limit, we may obtain the differential coefficient of nay function of one variable. In the same way, we may also derive the well-known rules for differentiating the product and quotient of two functions. It is assumed here that the reader is familiar with these ideas, and that the following examples will be sufficient to illustrate the technique of differentiating from first principles.

Example 2: The differential coefficient of y = sin x is obtained by evaluating

$$\frac{d(\sin x)}{dx} = \lim_{\delta x \to 0}\left(\frac{\sin(x+\delta x)-\sin x}{\delta x}\right) \tag{21}$$

$$= \lim_{\delta x \to 0}\left\{\frac{2\sin\frac{\delta x}{2}\cos\ x+\frac{\delta x}{2}}{\delta x}\right\} \tag{22}$$

$$= \lim_{\delta x \to 0}\left(\frac{\sin\frac{\delta x}{2}}{\delta x/2}\right) . \lim_{\delta x \to 0}\ \cos\ x+\frac{\delta x}{2} \tag{23}$$

(by Theorem 2, 3.1)

As $\delta x \to 0$, the first limit becomes equal to unity, and the second to cos x. Hence

$$\frac{d}{dx}(\sin x) = \cos x. \tag{24}$$

Example 3: If f and g are two functions of x, then

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}, \tag{25}$$

and
$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}. \tag{26}$$

Both of these well-known formulae can be proved from first principles, and we illustrate this statement by deriving (26).

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \lim_{\delta x \to 0}\left\{\frac{\frac{f(x+\delta x)}{g(x+\delta x)} - \frac{f(x)}{g(x)}}{\delta x}\right\} \tag{27}$$

$$= \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x g(x) - f(x)g(x+\delta x)}{g(x)g(x+\delta x)\delta x}\right\} \tag{28}$$

$$= \lim_{\delta x \to 0}\left\{\frac{1}{g(x)g(x+\delta x)} x\left[g(x).\frac{f(x+\delta x)-f(x)}{\delta x} - f(x)\frac{g(x+\delta x)-g(x)}{\delta x}\right]\right\} \tag{29}$$

which by using the theorems on limits stated in 3.1, and the definition of the differential coefficient, reduces to

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} \tag{30}$$

as required.

Example 4: The differential coefficients of the inverse circular functions $\sin^{-1}x$, $\cos^{-1}x$, (sometimes written as arc sin x, arc cos x) maybe obtained as follows:

If y = $\sin^{-1}x$, then x = sin y.

Hence $\frac{dx}{dy} = \cos y$ (31)

and $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}$ (32)

It is usual to take the positive sign of the square root in (32) to define the differential coefficient of the principal value of $\sin^{-1}x$, the principal value being such that $-\pi/2 \leqq \sin^{-1}x \leqq \pi/2$. When principal values of many-valued functions are implied it is usual to write the functions with capital letters. For example,

$\frac{d}{dx}(\mathrm{Sin}^{-1}x) = \frac{1}{\sqrt{(1-x^2)}}$ and $\frac{d}{dx}(\mathrm{Cos}^{-1}x) = -\frac{1}{\sqrt{(1-x^2)}}$,
where the principal value of $\cos^{-1}x$ is such that $0 \leqq \cos^{-1}x \leqq \pi$.

In the next chapter, we shall consider the operation of indefinite integration. This is the inverse operation to differentiation in that the differential coefficient of the indefinite integral of a function is the function itself.

SELF ASSESSMENT EXERCISES 3

I, Differentiate from the first principles.

a) $\sqrt{\left(\frac{x-1}{x+1}\right)}$

b) $\sqrt{a^2 - x^2}$

ii. Differentiate

a) $\log_e \cos \frac{1}{x}$

b) e^{3x^2}

c) $\sin^{-1} \frac{x}{x+1}$

d) $e^{\sin 2x}$

e) $x^{\cos x}$

3.4 Continuity and Differentiability

Continuity and differentiability are closely related in the sense that, if f(x) is a function of x and $\frac{df}{dx}$ exists at x = a, then f(x) is continuous at x = a. This follows since, if f(x) were not continuous at x = a, f(a + δ x $\rightarrow$ 0, and consequently

$$\lim_{\delta x \to 0} \left\{ \frac{f(a+\delta x) - f(a)}{\delta x} \right\} \qquad (33)$$

(which is the differential coefficient at x = a) could not exist. Hence, differentiability at a point implies continuity, whilst discontinuity implies non-differentiability. The converse, however, is not true; continuity does not imply differentiability. This maybe easily seen by considering the function represented graphically in Fig. 1.6. At the point P *(a,b)* the curve is continuous despite the 'kink' since the function is defined and the limit of the function as x$\rightarrow$ a from either direction is equal to f(a). The differential coefficient, however, is not uniquely defined at P *(a,b)* since a definite tangent to the curve at this point does not exist. The function is not differentiable therefore at this point, although (as shown) it is differentiable everywhere else. As an example, we mention the function f(x) = sin$\frac{1}{x}$, f(0) = 0, which is continuous at x = 0 but not differentiable there. Certain functions, moreover, are known to be continuous for all x and yet differentiable at none. Such functions are usually termed 'pathological' (i.e. ill) and are not often of any great interest in physical applications.

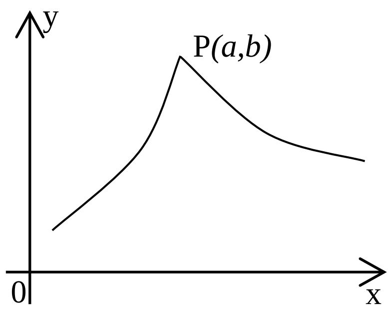


Fig. 1.5

3.5 Rolle's Theorem and the Mean-Value Theorem

i) Rolle's Theorem

if f(x) is continuous in the interval a $\leqq$ x $\leqq$ b and differentiable in a < x < b, and if f(a) = f(b) = 0, then, provided f(x) is not identically zero for a < x < b, there exists at least on value of x (say x = c) such that f'(c) = 0, where a < c < b. In words, there must exist at least one maximum or minimum in the interval *(a,b)*.

The validity of this theorem may be easily illustrated geometrically (see Fig. 1.7).

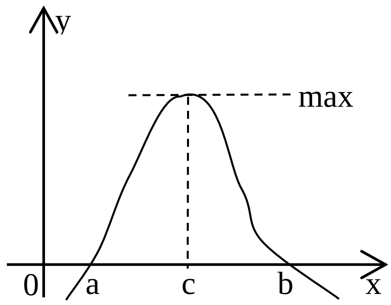


Fig. 1.6

ii) First Mean-Value Theorem

If f(x) is a continuous function of x in the interval a $\leqq$ x $\leqq$ b and is differentiable in a < x < b then there exists at least one value of x (say x = c) lying in the interval *(a,b)* such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{34}$$

In other words, considered graphically (see Fig. 3.8), there exists a value x = c such that the tangent to the curve at this point is parallel to the chord AB.

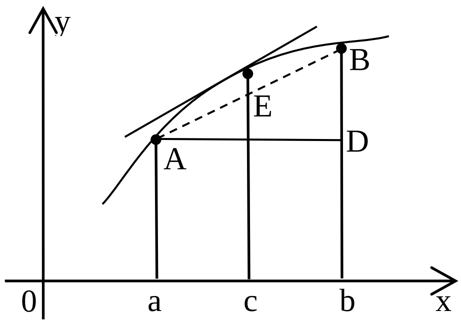


Fig. 1.7

We may prove this theorem geometrically in the following way: the equation of the line AB is

$$Y = f(a) + (x + a0\ \frac{f(b) - f(a)}{b - a}, \tag{35}$$

since BD = f(b) – f(a) and AD = b – a.

Hence the difference CE of the ordinates of the curve AB and the straight line AB is

$$F(x) = f(x) - y = f(x) - f(a) - (x - a)\ \frac{f(b) - f(a)}{b - a} \qquad (36)$$

Differentiating we have

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a} \qquad (37)$$

which is a defined quantity in $a < x < b$.

Also $F(a) = F(b) = 0$ since the curve AB and the straight line AB intersect at these points. Hence the function F(x) satisfies Rolle's Theorem and consequently there exists a value of x (say x = c) such that F'(c) = 0. This implies (from (37)) that there exists a value (x = c) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad (38)$$

which proves (34).

Example 5: If f(x) = sin 3x, and a = 0, b = $\pi/6$, c can be found from the equation (see (34) or (38))

$$3 \cos 3c = \frac{\sin(\pi/2) - \sin 0}{(\pi/6) - 0}. \qquad (39)$$

This gives directly $c = \frac{1}{3} \cos^{-1} (2/\pi)$.

The first Mean-Value Theorem is useful in many ways; in particular in establishing inequalities between functions. For example, a typical problem would be to show that in the interval $0 < x < \pi/2$

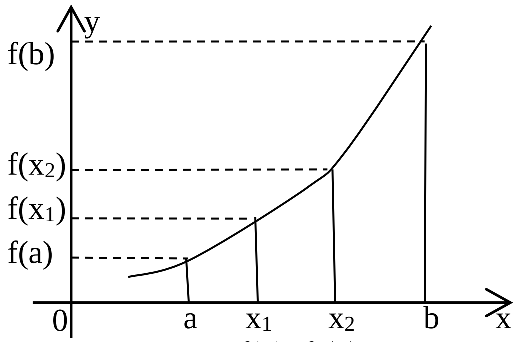
$$1 > \frac{\sin x}{x} > \frac{2}{\pi}. \qquad (40)$$

This is an extension of the inequality relation already obtained graphically in Chapter 2. Problems like this maybe conveniently dealt with by using the following result:

If f(x) is continuous in the range $a \leqq x \leqq b$, and differentiable in $a < x < b$, and if $f'(x) > 0$ in $a < x < b$, then for $a < x_1 < x_2 < b$
$f(a) < f(x_1) < f(x_2) < f(b)$.

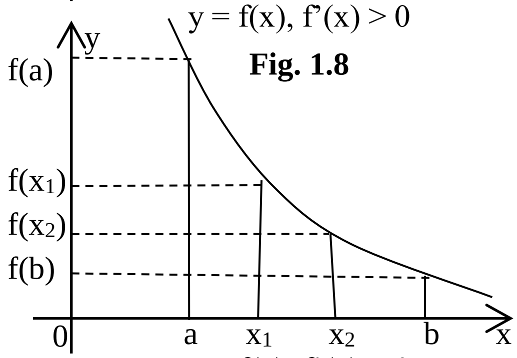
Similarly if $f'(x) < 0$ in $a < x < b$, then
$f(a) > f(x_1) > f(x_2) > f(b)$.
for $a < x_1 < x_2 < b$.

These statements are obvious when represented graphically (see Figs. 1.9 and 1.10), but we indicate an analytical proof here.



y = f(x), f'(x) > 0

Fig. 1.8



y = f(x), f'(x) < 0

Fig. 1.9

Consider the case when $f'(x) > 0$. The first Mean-Value Theorem gives

$$f(x_1) - f(a) = (x_1 - a)\, f'(c), \tag{41}$$

Where $a < c < x_1$. But if $f'(x) > 0$, then $f'(c) > 0$. Also, by assumption, $x_1 > a$. Hence

$$f(x_1) > f(a) \tag{42}$$

Similarly $f(x_2) > f(x_1)$ and $f(b) > f(x_2)$, and hence the statement is proved. A similar proof exists when $f'(x) < 0$.

Example 6: Consider now the inequality relation (40). Here

$$f(x) = \frac{\sin x}{x}, \text{ and } f(x) \to 1 \text{ as } x \to 0.$$

Differentiating we have

$$f'(x) = \frac{x\cos x - \sin x}{x^2}, \tag{43}$$

which is negative in the range $0 < x < \pi/2$. Hence according to the results obtained above we have

$$f(0) > f(x) > f(\pi/2), \tag{44}$$

whih gives $1 > \dfrac{\sin x}{x} > \dfrac{2}{\pi}$. (45)

3.6 Higher Derivatives and Leibnitz's Formula

When a function y = f(x) is differentiated more than once with respect to x, the higher differential coefficients are written as

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right),\ \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right),\ \ldots,\ \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right),$$

where $\dfrac{d^ny}{dx^n}$ is the nth differential coefficient of y with respect to x. (These are sometimes abbreviated to either

$f''(x), f'''(x) \ldots f^{(n)}(x)$

or $D^2y, D^3y \ldots. D^ny$,

where $D \equiv d/dx$.)

We now give a few examples showing how the nth differential coefficients of some simple functions maybe obtained.

Example 7: If y = sin x, then

$$Dy \equiv \frac{dy}{dx} = \cos x = \sin \ \frac{\pi}{2}+x\ ,$$

$$D^2y \equiv \left(\frac{d^2y}{dx^2}\right) = -\sin x = \sin(\pi + x),$$

$$D^3y \equiv \frac{d^3y}{dx^3} = -\cos x = \sin \ \frac{3\pi}{2}+x\ ,$$

and in general

$$D^ny \equiv \frac{d^ny}{dx^n} = \sin \ \frac{n\pi}{2}+x\ . \tag{46}$$

Example 8: If $y = \log_e x$, then

$$Dy = 1/x,$$

$$D^2y = 1/x^2,$$

$$D^3y = 2/x^3,$$

and $$D^n y = (-1)^{n-1} \frac{(n-1)!}{x^n}. \qquad (47)$$

(Equation (47) is valid for all n, including n = 1, if by 0! We mean unity.)

In these two examples, the functions have been simple enough to enable the nth differential coefficient to be written down in a few lines. When, however, the nth differential coefficient of a product of two functions u(x) and v(x) is required it is better to proceed as follows:

We have shown earlier from first principles that

$$D(uv) = u\,Dv + v\,Du \qquad (48)$$

Differentiating (48) now gives

$$D^2(uv) = u\,D^2v + 2Du\,.\,Dv + v\,D^2u. \qquad (49)$$

Similarly we obtain

$$D^3(uv) = u\,D^3v + 3Du\,.\,D^2v + 3D^2u\,.\,Dv + v\,D^3u, \qquad (50)$$

$$D^4(uv) = u\,D^4v+4Du\,.\,D^3v+6D^2u\,.D^2v+4D^3u\,.Dv+Dv + v\,D^4u \qquad (51)$$

and so on.

By inspection of these results the following formula (due to Leibnitz) may be written down for the nth differential coefficient of uv:

$$D^n(uv) = u\,D^nv + {}^nC_1Du\,.\,D^{n-1}v + {}^nC_2D^2u\,.\,D^{n-2}v + \ldots + {}^nC_{n-1}D^{n-1}u.Dv + v\,D^nu, \qquad (52)$$

where $^nC_r = \dfrac{n!}{(n-r)!r!}$.

This maybe written more concisely as

$$D^n(uv) = \sum_{r=0}^{n} {}^nC_r D^{n-r}v \,.\, D^n u. \qquad (53)$$

Leibnitz's formula (52) may be proved by induction as follows. Suppose (52) is true for one value of n, say m; then by differentiating we find

$$D^{m+1}(uv) = (u\,D^{m+1}v + Du \,.\, D^m v) + {}^mC_1(Du \,.\, D^m v + D^2u \,.\, D^{m-1}v)$$

$$+\, {}^mC_2(D^2u \,.\, D^{m-1}v + D^3u \,.\, D^{m-2}v) + \ldots + (Dv \,.\, D^m u + vD^{m+1}u), \qquad (54)$$

$$= uD^{m+1}v + (1+{}^mC_1)Du \,.\, D^m v + ({}^mC_1 + {}^mC_2)D^2u \,.\, D^{m-1}v + \ldots + vD^{m+1}u. \qquad (55)$$

Now ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$ (56)

and hence (55) becomes

$$D^{m+1}(uv) = uD^{m+1}v + {}^{m+1}C_1 Du \,.\, D^m v + {}^{m+1}C_2 D^2u \,.\, D^{m-1}v + \ldots + vD^{m+1}u. \qquad (57)$$

This again is the Leibnitz's formula (52) with m + 1 in place of m. Hence if the formula is true for n = m, it is certainly true for n = m + 1.

However, we know (from first principles) that it is true for n = 1, and therefore it is true for n =2, 3, …, and consequently for all positive integral values of n.

Example 9: To obtain the nth differential coefficient of $y = (x^2 + 1)e^{2x}$ we put $x^2 + 1 = u$ and $e^{2x} = v$. Then by (52)

$$(x^2 + 1)2^n e^{2x} + 2nx \,.\, 2^{n-1}e^{2x} + n(n-1)2^{n-2}e^{2x} \qquad (58)$$

$$= 2^{n-2}\, e^{2x}\,(4x^2 + 4nx + n^2 - n + 4). \qquad (59)$$

Example 10: The nth differential coefficient of $y = x\log_e x$ may be obtained by putting $x = u$ and $\log_e x$. Equation (52) then gives

$$D^n y = D^n(uv) = x(-1)^{n-1}\frac{(n-1)!}{x^n} + n(-1)^{n-1}\frac{(n-2)!}{x^{n-1}} \qquad (60)$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}, \ (n \geqq 2). \qquad (61)$$

Example 11: Leibnitz's formula may also be applied to a differential equation to obtain a relation between successive differential coefficients. As this forms a step towards finding power series solutions of certain types of differential equations now consider the following problem. Suppose y satisfies the equation

$$\frac{d^2y}{dx^2} + x^2y = \sin x. \tag{62}$$

Then differentiating each term n times (using Leibnitz's formula for the product term x^2y), we obtain (using (46))

$$D^{n+2}y + (x^2D^ny + 2nx\, D^{n-1}y + n(n-1)D^{n-2}y) = \sin\left(\frac{n\pi}{2} + x\right), \tag{63}$$

which is a relation between the (n – 2)th, (n – 1)th, nth and (n + 2)th differential coefficients of y for all x. If we now put x = 0 in (63) we find

$$D^{n+2}y + n(n-1)D^{n-2}y = \begin{cases} 0 \text{ if n is even}, \\ \pm 1 \text{ if n is odd}. \end{cases} \tag{64}$$

Remark: Relations of the type of the expression in (63) and (64) at x = 0, are useful in developing power series solutions of differential equations.

SELF ASSESSMENT EXERCISES 4

1) If y is a function of x, show by putting $\frac{dy}{dx}$ = p, that $\frac{d^2x}{dy^2}$ =

$$-\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}.$$

2) If $y = \sin \pi\sqrt{(x+1)}$ prove that $4(x+1)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \pi^2 y = 0$.

3.7 Maxima and Minima

A particular important application of differentiation is to the problem of finding the maxima and minima values of a given function f(x) in same interval a $\leqq$ x $\leqq$ b. Purely on geometrical grounds we can see (Fig. 3.11) that provided f(x) is differentiable in the range *(a,b)* then at a maximum or minimum the tangent to the curve must be parallel to the x-axis. According a necessary condition for a point x_0 (say) to be a maximum or a minimum is that

$$f'(x_0) = 0 \left(\text{i.e.} \frac{df(x)}{dx} = 0 \text{ at } x = x_0 \right). \quad (65)$$

Such points are called critical points.

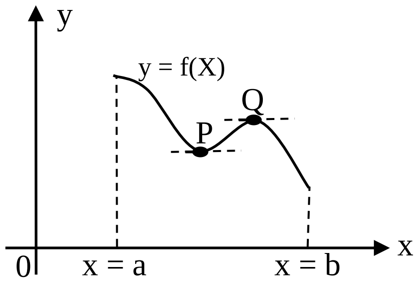


Fig. 1.10

Although at the end points x = a and x=b it would seem that the function possesses larger and smaller values respectively than at the maximum point Q and the minimum point P, we do not count these as true maxima and minima but note that they are just the greatest and least values of the function in the range $a \leqq x \leqq b$.

Now suppose $f'(x) > 0$. Then the function $y = f(x)$ increases with increasing x. if $f''(x) > 0$ then $f'(x)$ is also increasing and hence the curve is concave is upwards (as near the maximum point Q). Hence if $f'(x_0) = 0$ and $f''(x_0) > 0$ the point x_0 is a minimum point, whilst if $f'(x_0) = 0$ and $f''(x_0) < 0$ the point x_0 is a maximum point. It may happen that both $f'(x_0)$ and $f''(x_0)$ vanish (for example, $f(x) = x^3$ has a critical point at $x = x_0 = 0$, and $f''(0) = 0$). Such points are called points of inflection. A more detailed theory based on Taylor series (see Unit4) enables the nature of a critical point to be determined when the first n (say) derivatives vanish at the critical point. However, we shall not deal with this situation here.

Finally, we note that we have so far assumed that the function is continuous and has a continuous first derivative. If the function is not differentiable then it may still possess maxima and minima but they cannot be found by differentiation. For example, the function $y = |x|$ is shown in Fig. 3.12. This is not a differentiable function for the range $-a \leqq x \leqq a$ (say). However, a true minimum does exist at $x = 0$.

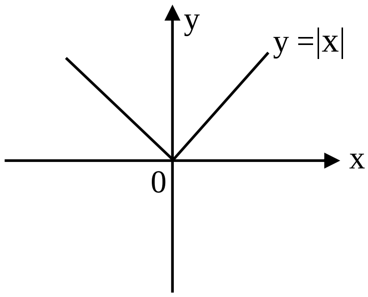


Fig. 1.11

SELF ASSESSMENT EXERCISES 5

i. Determine the maxima and minima value (if any) of

a) $\sin^{-1}(x^2 + 2)$

b) $1 + x^{2/3}$

ii. Find the critical point of $y = x^2e^{-x}$ and determine whether they are maxima or minima.

4.0 CONCLUSION

In this unit, we have dealt with limit, continuity and have established some theorems, such as Rolle's Theorem and Mean-Valued theorem.

We have also established relationship between continuity and differentiability.

The concept of differentiability allows us to determine, the minimum and maximum point of a given function.

5.0 SUMMARY

Here you have learnt about limits, continuity and differentiability. You have also learnt that differentiability at a point implies continuity at that point.

Some relevant theorems such as Rolles and Mean-valued theorem were also studied.

You are to master these areas in order to be able to follow the presentation in the next unit.

6.0 TUTOR-MARKED ASSIGNMENT

i. Find the derivatives of

a) $\dfrac{1}{\sqrt{(ax+b)}}$

b) $x^4 \log_e x$

ii. Find the points of discontinuity of the following functions

a) $\dfrac{4x+6}{x^2-6x+8}$

b) sec x

c) $\dfrac{\sin x}{\sqrt{x}}$

iii. If $y = \sin \pi\sqrt{(x+1)}$ prove that $4(x + 1)\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + \pi^2 y = 0$.

iv. If $y = \sin^{-1}x = (\sin^{-1}x)^2$, prove that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx}$ is independent of x.

7.0 REFERENCES/FURTHER READING

G. S. Stephenson: Mathematical Methods for Science Students. Longman, London and New York, 1977.

P. D. S. Verma; Engineering Mathematics Vikas Publishing House, PVT Ltd., New Delhi, 1995.

UNIT 2 PARTIAL DIFFERENTIATION

CONTENTS

1.0 INTRODUCTION

In Unit 1, we discussed the concepts of continuity and differentiability of one real independent variable. In this unit, we shall consider and extend the idea developed in Unit 1 to function of more than one variable.

This is very important because, in scientific analysis of a problem, one often find that a factor depends upon several other factors. For example, volume of a solid depends upon its length, breath and height. Strength of a material depends upon temperature, density, isotropy, softness etc.

It is therefore necessary to define function of several variables. If a variable z depends for its value upon those of x and y, we say z is a function of x and y, and write $z = f(x,y)$.

All types of concepts for functions of one variable are extend to functions of several variables. For example the value of a function f(x,y) at (x_0, y_0) is given by $f(x_0, y_0)$. The domain and range of the function are defined as before.

2.0 OBJECTIVES

After studying this unit you should be able to:

- relate the concepts of limit and continuity studied in unit 1 to function of several variables
- carry out partial differentiation of function of several variables
- apply the concept of Lagrange multiplier techniques to finding the minima and maxima of functions of several variables
- find higher derivatives of functions of several variables
- carry out Taylor series expansion of functions of several variables.

3.0 MAIN CONTENT

3.1 Functions of Several Independent Variables

The concepts of continuity and differentiability of functions of one real independent variable have already been discussed in Unit 1 and in this section we extend these ideas to functions of two or more real independent variables x, y, u, v … (or x_1, x_2, x_3 …). We first discuss, however, some general properties of functions of this type.

Consider, for example, a function of two variables x and y defined by

$$f(x,y) = x^2 - 2y^2. \qquad (1)$$

Then the value of *f(x,y)* s determined by (1) for every number pair *(x,y)*. For instance, if *(x, y)* = (0, 0) we have f(0,0) = 0 and if *(x ,y)* = (1, 0), f(1, 0) = 1.

In general we may represent every pair of numbers *(x,y)* by a point P in the *(x,y)* plane of a rectangular Cartesian coordinate system and denote the corresponding value of *f(x,y)* by the length of the line PP' drawn parallel to the z=axis (see Fig 2.1). The locus of all points such as P' is then a surface in the (x, y, z) space which represents the function *f(x,y)*. However, this simple geometrical picture is impossible to

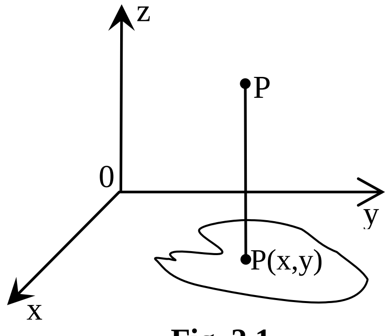


Fig. 2.1

visualise when dealing with functions of three or more independent variables.

Returning now for simplicity to functions of two independent variables, we notice that many functions are only defined within a certain region of the *(x,y)* plane. (This is analogous to the one-variable case where f(x) is defined in a certain interval of x).
For example, the real function.

$$f(x,y) = \sqrt{a^2 - x^2 - y^2} \qquad (2)$$

is only defined within and on the boundary of the circle $x^2 + y^2 = a^2$; outside this region it takes on imaginary values. Similarly the function

$$f(x,y) = \tan \frac{y}{x} \qquad (3)$$

is undefined along the line x = 0. The function given in (1), however, is defined for all values of x and y. it is usual to denote the region of definition of a function of several independent variables by the letter.

If a function *f(x,y)* has just one real value for every *(x,y)* value within its region of definition R, we say that it is a single-valued function are obtained for a given *(x,y)* value we call the function two-valued or many-valued. For instance, the function defined by (1) is single-valued over the region R given by $-\infty < x < \infty$, $-\infty < y < \infty$, whereas the function defined by (2) is two-valued over the region R given by $x^2 + y^2 < a^2$ (since both signs of the square root may be taken) and single-valued (equal to zero) on the boundary of the circle $x^2 + y^2 = a^2$.

Another important concept already defined in Unit 1, 3.2 for functions of one independent variable is that of continuity. When discussing the continuity of functions of two or more independent variables similar considerations apply. Suppose *f(x,y)* is a real single-valued function of x and y. Then if *f(x,y)* approaches a value l as x approaches a and y approaches b, l is said to be a limit of *f(x,y)* as the point *(x,y)* approaches the point *(a,b)* and is written as

$$\lim_{(x,y)\to(a,b)} f(x,y) = l. \qquad (4)$$

However, as we have already seen in one-variable case, x may approach a specified point x = a from either the negative side ($-\infty \to a$) or from the positive side (-$\infty \to$ a), and the values of the two limits so obtained may be different. The same is true of (4); the way in which *(x,y)* $\to$ *(a,b)* may determine the value of l. However, there is now much more freedom than in the one-variable case since (see Fig. 9.2) the point Q*(x,y)* may approach the point P*(a,b)* along any of the infinity of curves, say c, which lie in the *(x,y)* plane and which pass through P. If, however, the limit exists independently of the way in which Q approaches P and is such that

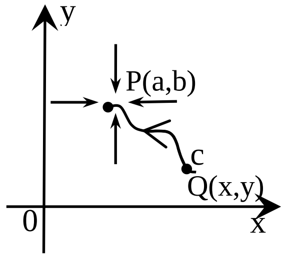


Fig. 2.2

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b), \qquad (5)$$

(assuming that *f(a,b)* exists), then *f(x,y)* is said to be a continuous function of x and y at the point *(a,b)*. Likewise, if a function *f(x,y)* is continuous at every point of a region R of the *(x,y)* plane it is said to be continuous over that region.

3.2 First Partial Derivatives

Suppose *f(x,y)* is a real single-valued function of two independent variables x and y. Then the partial derivatives of *f(x,y)* with respect to x is defined as

$$\frac{\partial f}{\partial x}_y = \lim_{\delta x\to 0}\left\{\frac{f(x+\delta x, y)-f(x,y)}{\delta x}\right\} \qquad (6)$$

Similarly the partial derivative of *f(x,y)* with respect to y is defined as

$$\frac{\partial f}{\partial x}_x = \lim_{\delta x\to 0}\left\{\frac{f(x, y+\delta y)-f(x,y)}{\delta y}\right\} \qquad (7)$$

In other words the partial derivative of *f(x,y)* with respect to x may be thought of a s the ordinary derivative of *f(x,y)* with respect to x obtained by treating y as a constant. Similarly, the partial derivative of *f(x,y)* with respect to y may be found by treating x a s constant and evaluating the ordinary derivative of *f(x,y)* with respect to y. The variable, which is to be held constant in the differentiation, is denoted by a subscript as shown in (6) and (7). Alternative notations, however, exist for partial derivatives and one of the more useful and compact of these is to denote $\frac{\partial f}{\partial x}_y$ by f_x, and $\left(\frac{\partial f}{\partial y}\right)_x$ by f_y.

The subscripts appearing in the f now denote the variables with respect to which *f(x,y)* is to be differentiated.

The following examples illustrate the evaluation of first partial derivatives.

Example 1: If

$$f(x,y) = x^2 - 2y^2 \tag{8}$$

(see (1)), then

$$f_x = \frac{\partial f}{\partial x}_y = \lim_{\delta x \to 0}\left\{\frac{[(x+\delta x)^2 - 2y^2] - (x^2 - 2y^2)}{\delta x}\right\} \tag{9}$$

$$= \lim_{\delta x \to 0}\left(\frac{2x\delta x + (\delta x)^2}{\delta x}\right) = 2x \tag{10}$$

Similarly

$$f_x = \left(\frac{\partial f}{\partial y}\right)_x = \lim_{\delta x \to 0}\left\{\frac{[x^2 - 2(y+\delta y)^2] - (x^2 - 2y^2)}{\delta y}\right\} \tag{11}$$

$$= \lim_{\delta x \to 0}\left(\frac{-4y\delta y - 2(\delta y)^2}{\delta y}\right) = -4y \tag{12}$$

Example 2: The last example illustrated the technique of partial differentiation from first principles (i.e. by the evaluation of a limit). We now differentiate partially by keeping certain variables constant as required. For example, if

$$f(x,y) = \sin^2 x \cos y + \frac{x}{y^2}, \tag{13}$$

then keeping y constant we find

$$f_x = \frac{\partial f}{\partial x}_y = 2 \sin x \cos x \cos y + \frac{1}{y^2}. \tag{14}$$

Similarly, keeping x constant,

$$f_y = \frac{\partial f}{\partial x}_x = 2 \sin^2 x \sin y - \frac{2x}{y^3}. \tag{15}$$

Example 3: To obtain the partial derivatives of a function of n independent variables any n -1 of these variables must be held constant and the differentiation carried out with respect to the remaining variable.

There are therefore n first partial derivatives of such a function. For Example, if

$$f(x, y, z) = e^{2z} \cos xy \qquad (16)$$

SELF ASSESSMENT EXERCISE 1

Given that $f(x,y) = x^2y + \sin^{-1}x$ find

i) f_x

ii) f_y

then

$$f_x = \frac{\partial f}{\partial x}_y = -ye^{2z} \sin xy, \qquad (17)$$

$$f_y = \left(\frac{\partial f}{\partial y}\right)_{x.z} = xe^{2z} \sin xy, \qquad (18)$$

and

$$f_z = \frac{\partial f}{\partial z}_{x,y} = 2e^{2z} \cos xy \qquad (19)$$

3.3 Function of a Function

It is a well-known property of functions of one independent variable that if f is a function of a variable u, and u is a function of a variable x, then

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}. \qquad (20)$$

This result may be immediately extended to the case when f is a function of two or more independent variables. Suppose f = f(u) and u = u*(x,y)*. Then, by the definition of a partial derivative,

$$f_x = \frac{\partial f}{\partial x}_y = \frac{df}{du}\frac{\partial u}{\partial x}_y, \qquad (21)$$

$$f_y = \left(\frac{\partial f}{\partial y}\right)_x = \frac{df}{du}\left(\frac{\partial u}{\partial y}\right)_x. \qquad (22)$$

Example 4: If

$$f(x,y) = \tan^{-1}\frac{y}{x} \tag{23}$$

then putting u = y/x we have

$$f_x = \frac{\partial f}{\partial x}_y = \frac{d}{du}(\tan^{-1}u)\ \frac{\partial u}{\partial x}_y = \frac{y}{x^2+y^2} \tag{24}$$

and

$$f_y = \left(\frac{\partial f}{\partial y}\right)_x = \frac{d}{du}(\tan^{-1}u)\left(\frac{\partial u}{\partial y}\right)_x = \frac{x}{x^2+y^2} \tag{25}$$

Example 5: If f(u) = sin u and u = $\sqrt{(x^2+y^2)}$ then

$$f_x = \frac{\partial f}{\partial x}_y = (\cos u)\ \frac{x}{\sqrt{(x^2+y^2)}} = \frac{x\cos\sqrt{(x^2+y^2)}}{\sqrt{(x^2+y^2)}}, \tag{26}$$

and

$$f_y = \left(\frac{\partial f}{\partial y}\right)_x = (\cos u)\ \frac{y}{\sqrt{(x^2+y^2)}} = \frac{y\cos\sqrt{(x^2+y^2)}}{\sqrt{(x^2+y^2)}}, \tag{27}$$

3.4 Higher Partial Derivatives

Provided the first partial derivatives of a function are differentiable, we may differentiate them partially to obtain the second partial derivatives. The four second partial derivatives of *f(x,y)* are therefore

$$f_{xy} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}f_x = \frac{\partial}{\partial x}\ \frac{\partial f}{\partial x}_y, \tag{28}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}f_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)_x, \tag{29}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x\partial y^2} = \frac{\partial}{\partial x}f_y = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_x, \tag{30}$$

and

$$f_{yx} = \frac{\partial^2 f}{\partial y\partial x^2} = \frac{\partial}{\partial y}f_x = \frac{\partial}{\partial y}\ \frac{\partial f}{\partial x}_y, \tag{31}$$

Higher partial derivatives than the second maybe obtained in a similar way.

Example 6: We have already seen in Example 4 that if

$$f(x,y) = \tan^{-1}\frac{y}{x} \tag{32}$$

then

$$\frac{\partial f}{\partial x}_{y} = -\frac{y}{\sqrt{(x^2+y^2)}},\ \left(\frac{\partial f}{\partial y}\right)_x = \frac{x}{\sqrt{(x^2+y^2)}}. \tag{33}$$

Hence, differentiating these first derivatives partially, we obtain

$$f_{xy} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2} \tag{34}$$

and

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = -\frac{2xy}{(x^2+y^2)^2} \tag{35}$$

Also

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2+x^2}{(x^2+y^2)^2} \tag{36}$$

and

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} \tag{37}$$

Since (36) and (37) are equal we have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \tag{38}$$

which shows that the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are commutative. We shall return to this point in the next section. Finally we note that if (34) and (35) are added then *f(x,y)* satisfies the partial differential equation (Laplace's equation in two variables)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \quad (39)$$

In general, any function satisfying this equation is called a harmonic function.

3.3.1 Commutative Property of Partial Differentiation

In Example 6 we have shown that the second partial derivatives f_{xy} and f_{yx} of the function $f(x,y) = \tan^{-1}\frac{y}{x}$ are equal. This is in fact the case for most functions as can be verified by choosing a few functions at random. It can be proved that a sufficient (but not necessary) condition that $f_{xy} = f_{yx}$ at some point *(a,b)* is that both f_{xy} and f_{yx} are continuous at *(a,b)* and in all that follows it will be assumed that this condition is satisfied.

SELF ASSESSMENT EXERCISES 2

Show that $f_{xy} = f_{yx}$ for the following functions

i. $f(x,y) = x^2 - xy + y^2$

ii. $f(x,y) = x \sin (y - x)$

iii. $f(x,y) = e^y \log_e (x + y)$

iv. $f(x,y) = \dfrac{xy}{x^2 + y^2}$

3.4 Total Derivatives

Suppose $f(x,y)$ is a continuous function defined in a region R of the xy-plane, and that both $\frac{\partial f}{\partial x}_y$ and $\frac{\partial f}{\partial x}_y$ are continuous in this region. We now consider the change in the value of the function brought about by following small changes in x and y.

If δf is the change in due to change δx and δy in x and y then

$$\delta f = f(x + \delta x, y + \delta y) - f(x,y) \quad (40)$$

$$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x,y). \quad (41)$$

Now by definition (see (6) and (7))

$$\frac{\partial}{\partial x} f(x, y + \delta y) = \lim_{\delta x \to 0} \left\{ \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \right\} \quad (42)$$

and

$$\frac{\partial}{\partial y} f(x,y) = \lim_{\delta x \to 0} \left\{ \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \right\} \quad (43)$$

Consequently

$$f(x+\delta x, y+\delta y) - f(x, y+\delta y) = \left[\frac{\partial}{\partial x} f(x, y+\delta y) + \alpha \right] \delta x, \quad (44)$$

and

$$f(x, y + \delta y) - f(x,y) = \left[\frac{\partial}{\partial y} f(x, y) + \beta \right] \delta y, \quad (45)$$

where α and β satisfy the conditions

$$\lim_{\delta x \to 0} \alpha = 0 \text{ and } \lim_{\delta x \to 0} \beta = 0. \quad (46)$$

Using (44) and (45) in (41) we

$$\delta f = \left[\frac{\partial}{\partial x} f(x, y+\delta y) + \alpha \right] \delta x + \left[\frac{\partial}{\partial y} f(x, y) + \beta \right] \delta y. \quad (47)$$

Furthermore, since all first derivatives are continuous by assumption, the first term of (47) may be written as

$$\frac{\partial}{\partial x} f(x, y + \delta y) = \frac{\partial f(x,y)}{\partial x} + y, \quad (48)$$

where γ satisfies the condition

$$\lim_{\delta x \to 0} \gamma = 0 \quad (49)$$

Hence, using (48), (47) now becomes

$$\delta f = \frac{\partial f(x,y)}{\partial x} \delta x + \frac{\partial f(x,y)}{\partial y} \delta y + (\alpha + \gamma) \delta x + \beta \delta y. \quad (50)$$

The expression

$$\delta f \cong \frac{\partial f(x,y)}{\partial x} \delta x + \frac{\partial f(x,y)}{\partial y} \delta y \quad (51)$$

obtained by neglecting the small terms $(\alpha + \delta)\delta x$ and $\beta\delta y$ in (50) represents, to the first order in δx and δy, the change in *f(x,y)* due to changes δx and δy in x and y respectively.

It is easily seen that the first term of (51) represents the change in *f(x,y)* due to a change δx in x keeping y constant; similarly the second term is the change in *f(x,y)* due to a change δy in y keeping x constant. The total differential is nothing more than the sum of these two effects. In the case of a function of n independent variables $f(x_1, x_2, \dots x_n)$ we have

$$\delta f \cong \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \dots + \frac{\partial f}{\partial x_n}\delta x_n = \sum_{r=1}^{n}\frac{\partial f}{\partial x_r}\delta x_r. \qquad (52)$$

The following examples illustrate the use of these results.

Example 7: To find the change in

$$f(x,y) = xe^{xy} \qquad (53)$$

when the values of x and y are slightly changed from 1 and 0 to $1 + \delta x$ and δy respectively. We first use (51) to obtain

$$\delta f \cong (xye^{xy} + e^{xy})\,\delta x + x^2e^{xy}\delta y. \qquad (54)$$

Hence putting x =1, y = 0 in (54) we have

$$\delta f \cong \delta x + \delta y. \qquad (55)$$

For example, if $\delta x = 0.10$ and $\delta y = 0.05$, then $\delta f \cong 0.15$.

We now return to the exact expression for δf given in (50). Suppose u = *f(x,y)* and that both x and y are differentiable functions of a variable t so that

$$x = x(t),\ y = y(t) \qquad (56)$$

and

$$u = u(t) \qquad (57)$$

Hence dividing (50) by δt and proceeding to the limit $\delta t \to 0$ (which implies $\delta x \to 0$, $\delta y \to 0$ and consequently $\alpha, \beta, \delta \to 0$) we have

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}. \qquad (58)$$

This expression is called the total derivative of *u(t)* with respect to t. It is easily seen that if

$u = f(x_1, x_2, x_3 \dots x_n),$ (59)

Where $x_1, x_2, x_3 \dots x_n$ are all differentiable functions of a variable t, then u = *u(t)* and

$$\frac{du}{dt} = \frac{\partial f}{\partial x_1}.\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}.\frac{dx_2}{dt} + \dots + \frac{\partial f}{\partial x_n}.\frac{dx_n}{dt} = \sum_{r=1}^{n} \frac{\partial f}{\partial x_r}.\frac{dx_r}{dt}. \qquad (60)$$

Example 8: Suppose

$u = f(x,y) = x^2 + y^2$ (61)

and

$x = \sinh t,\ y = t^2.$ (62)

Then by direct substitution we have

$u(t) = \sinh^2 t + t^4$ (63)

and consequently

$$\frac{du}{dt} = 2\sinh^2 t \cosh t + t^4 \qquad (64)$$

We now obtain this result using the expression for the total derivative. Since

$$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y, \qquad (65)$$

$$\frac{dx}{dt} = \cosh t,\ \frac{dy}{dt} = 2t, \qquad (66)$$

(58) gives

$$\frac{du}{dt} = 2x \cosh t + 4yt \qquad (67)$$

$= 2\sinh t \cosh t + 4t^3,$ (68)

as before.

3.5 Implicit Differentiation

A special case of the total derivative (580 arises when y is itself a function of x (i.e. t = x). Consequently u is a function of x only and

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}. \tag{69}$$

Example 9: Suppose

$$u = f(x,y) = \tan^{-1}\frac{x}{y} \tag{70}$$

and

$$y = \sin x. \tag{71}$$

Then by (69) we have

$$\frac{du}{dx} = \frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}\cos x \tag{72}$$

$$= \frac{\sin x - x\cos x}{x^2+\sin^2 x} \tag{73}$$

This result could have been obtained by the slightly more laborious method of substituting (71) into (70) and then differentiating with respect to x in the usual way.

When y is defined as a function of x by the equation

$$u = f(x,y) = 0 \tag{74}$$

y is called an implicit function of x. since u is identically zero its total derivative must vanish, and consequently from (69)

$$\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right)_y \Big/ \left(\frac{\partial f}{\partial y}\right)_x. \tag{75}$$

Example 10: The gradient of the tangent at any point *(x,y)* of the conic

$$f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \tag{76}$$

(where a, h, b, g, f and c are constants) is, by (75),

$$\frac{dy}{dx} = -\frac{2ax + 2hy + 2g}{2by + 2hx + 2f}. \tag{77}$$

Example 11: The pair of equations

$$F(x, y, z) = 0, \; G(x, y, z) = 0, \tag{78}$$

Where F and G are differentiable functions of x, y and z define, for example, y and z as functions of x. Hence, since the total derivatives of F(x, y, z) and G(x, y, z) are identically zero, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}.\frac{dy}{dx} + \frac{\partial F}{\partial z}.\frac{dz}{dx} = 0 \tag{79}$$

and

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}.\frac{dy}{dx} + \frac{\partial G}{\partial z}.\frac{dz}{dx} = 0, \tag{80}$$

whence

$$\frac{dy}{dx} = - \; \frac{\partial F}{\partial x}.\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}.\frac{\partial G}{\partial x} \; \left(\frac{\partial F}{\partial y}.\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}.\frac{\partial G}{\partial y}\right)^{-1} \tag{81}$$

and

$$\frac{dz}{dx} = \left(\frac{\partial F}{\partial x}.\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}.\frac{\partial G}{\partial x}\right)\left(\frac{\partial F}{\partial y}.\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}.\frac{\partial G}{\partial y}\right)^{-1} \tag{82}$$

For example, if

$$F(x, y, z) = x^2 + y^2 + z^2,$$

$$G(x, y, z) = x^2 - y^2 + 2z^2, \tag{83}$$

then

$$\frac{\partial F}{\partial x} = 2x, \; \frac{\partial F}{\partial y} = 2y, \; \frac{\partial F}{\partial z} = 2z, \tag{84}$$

$$\frac{\partial G}{\partial x} = 2x, \; \frac{\partial G}{\partial y} = -2y, \; \frac{\partial G}{\partial z} = 4z, \tag{85}$$

and hence, by (81) and (82),

$$\frac{dy}{dx} = \frac{x}{3y}, \; \frac{dz}{dx} = -\frac{2x}{3z} \tag{86}$$

3.6 Higher Total Derivatives

We have already seen that if u = *f(x,y)* and x and y are differentiable functions of t then

$$\frac{du}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt}. \tag{87}$$

To find $\frac{d^2u}{dt^2}$ we note from (87) that the operator $\frac{d}{dt}$ can be written as

$$\frac{d}{dt} \equiv \frac{dx}{dt}.\frac{\partial}{\partial x} + \frac{dy}{dt}.\frac{\partial}{\partial y} \tag{88}$$

Hence

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\frac{du}{dt} = \left(\frac{dx}{dt}.\frac{\partial}{\partial x} + \frac{dy}{dt}.\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt}\right) \tag{89}$$

$$= \frac{\partial^2 f}{\partial x^2}\frac{dx}{dt}^{2} + 2\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt}\left(\frac{dy}{dt}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial f}{\partial x}.\frac{d^2x}{dt^2} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} \tag{90}$$

where we have assumed that $f_{xy} = f_{yx}$. Higher total derivative may be obtained in similar way.

A special case of (90) which will be needed later is when

$$\frac{dx}{dt} = h, \ \frac{dy}{dt} = k, \tag{91}$$

where h and k are constants. We then have

$$\frac{d^2u}{dt^2} = h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}, \tag{92}$$

which, if we define the differential operator *D by

$$^{*}D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}, \tag{93}$$

may be written symbolically as

$$\frac{d^2u}{dt^2} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f = {}^{*}D^2 f. \tag{94}$$

Similarly we find

$$\frac{d^3u}{dt^3} = h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3} \quad (95)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f = {}^*D^3 f, \quad (96)$$

assuming the commutative property of partial differentiation. In general,

$$\frac{d^n u}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f = {}^*D^n f, \quad (97)$$

where the operator $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$ is to be expanded by means of the binomial theorem.

3.7 Homogeneous Functions

A function *f(x,y)* is said to be homogeneous of degree m if

$$f(kx, ky) = k^m f(x,y), \quad (98)$$

where k is a constant. A similar definition applies to a function of any number of independent variables. For example,

$$f(x,y) = x^3 + 4xy^2 - 3y^3 \quad (99)$$

is homogeneous of degree 3 since

$$(kx)^3 + 4(kx)(ky)^2 - 3(ky)^3 = k^3[x^3 + 4xy^2 - 3y^3]. \quad (100)$$

Similarly

$$f(x,y) = \frac{x^2y^2}{4xy} + \frac{y}{x}\sin\left(\frac{x}{y}\right) \quad (101)$$

is homogeneous of degree 0 since

$$\frac{(kx)^2 + (ky)^2}{4k^2xy} + \frac{ky}{kx}\sin\left(\frac{kx}{ky}\right) = k^0\left\{\frac{x^2+y^2}{4xy} + \frac{y}{x}\sin\frac{x}{y}\right\}. \quad (102)$$

3.8 Euler's Theorem

Theorem 1: If $u = f(x_1, x_2, \dots x_n)$ is a homogeneous differentiable function of degree m in the independent variables $x_1, x_2, \dots x_n$, then where k is a constant. Then since u is homogeneous $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \ldots + x_n\frac{\partial f}{\partial x_n} = mf$. (103)

To prove this theorem we define a new set of variables $y_1, y_2, \ldots y_n$ by the relations.

$x_1, = y_1k, \; x_2, = y_2k \ldots x_n = yn$ of degree m (104)

$u = f(y_1k, y_2k \ldots y_nk) = k^m f(y_1, y_2, \ldots y_n)$. (105)

Differentiating (105) with respect to k e find

$$\frac{du}{dk} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dk} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dk} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dk} = mk^{m-1}f(y_1, y_2, \ldots y_n) \qquad (106)$$

or

$$\frac{du}{dk} = y_1\frac{\partial f}{\partial x_1} + y_2\frac{\partial f}{\partial x_2} + \ldots + y_n\frac{\partial f}{\partial x_n} = mk^{m-1}f. \qquad (107)$$

Hence multiplying the last two expression of (107) by k we have

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \ldots + x_n\frac{\partial f}{\partial x_n} = mf, \qquad (108)$$

which proves the theorem.

Example 12: The function

$f(x,y) = x^3 + 4xy^2 - 3y^3$ (109)
is homogeneous of degree 3 and hence, by Euler's Theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \; 3f. \qquad (110)$$

This is easily verified since

$$\frac{\partial f}{\partial x} = 3x^2 + 4y^2, \; \frac{\partial f}{\partial y} = 8xy - 9y^2. \qquad (111)$$

Hence
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x(3x^2 + 4y^2) + y(8xy - 9y^2)$$

$$= 3(x^3 + 4xy^2 - 3y^3) = 3f. \qquad (112)$$

3.9 Change of Variables

We have seen earlier on in this chapter that if u = *f(x,y)* is a continuous and differentiable function of the independent variables x, y and if x and y are differentiable functions of a variable then

$$\frac{du}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt}. \qquad (113)$$

Suppose now that x and y are functions not just of one variable but of two, say s and t, such that

$$x = x(s, t), \quad y = y(s, t). \qquad (114)$$

Clearly since u is a function of x and y it is also a function of s and t and necessarily has the two partial derivatives $\frac{\partial u}{\partial s}_t$ and $\frac{\partial u}{\partial t}_s$. Hence keeping t a constant and differentiating with respect to s, we have (following (113))

$$\frac{\partial u}{\partial s}_t = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial s}_t + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial s}\right)_t. \qquad (115)$$

Similarly, keeping s a constant and differentiating with respect to t

$$\frac{\partial u}{\partial t}_s = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial t}_s + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial t}\right)_s. \qquad (116)$$

Example 13: Given that u = *f(x,y)* and

$$x = s^2 - t^2, \ \ y = 2st, \qquad (117)$$

prove that

$$s\frac{\partial u}{\partial s} - t\frac{\partial u}{\partial t} = 2(s^2 + t^2)\ \frac{\partial f}{\partial x}. \qquad (118)$$

From (115), (116) and (117) we have

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\ \frac{\partial y}{\partial s} = 2s\frac{\partial f}{\partial x} = 2\frac{\partial f}{\partial y}, \qquad (119)$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = -2t\frac{\partial f}{\partial x} + 2s\frac{\partial f}{\partial y}. \tag{120}$$

Hence multiplying (119) by s and (120) by t and subtracting we obtain (118) as required.

Example 14: Given u = *f(x,y)* and

$$x = r\cos\theta,\ y = r\sin\theta, \tag{121}$$

prove that

$$r\frac{\partial u}{\partial r} = x\frac{\partial f}{\partial x} = y\frac{\partial f}{\partial y} \tag{122}$$

and

$$\frac{\partial u}{\partial \theta} = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}. \tag{123}$$

These results are easily obtained since from (115) we have

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta, \tag{124}$$

which, on multiplying through by r, gives (122). Similarly from (116)

$$\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}r\cos\theta \tag{125}$$

$$= x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}, \tag{126}$$

which is (123).

Example 15: If x and y are rectangular Cartesian coordinates and if u = *f(x,y)* satisfies Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, \tag{127}$$

obtain the form of this equation in polar coordinates (r, θ), where x = r $\cos\theta$, y = r sin θ.

From (115) and (116) we have

$$\frac{\partial u}{\partial r}_{\theta} = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial r}_{\theta} + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial r}\right)_{\theta} \tag{129}$$

$$= \cos\theta\,\frac{\partial f}{\partial x} + \sin\theta\,\frac{\partial f}{\partial y}, \tag{130}$$

and

$$\frac{\partial u}{\partial \theta}_{r} = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial \theta}_{r} + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial \theta}\right)_{r} \tag{131}$$

$$= -r\sin\theta\,\frac{\partial f}{\partial x} + r\cos\theta\,\frac{\partial f}{\partial y}. \tag{132}$$

Solving (130) and (132) for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ we find

$$\frac{\partial f}{\partial x} = \cos\theta\,\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial u}{\partial \theta} \tag{133}$$

$$\frac{\partial f}{\partial y} = \sin\theta\,\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial u}{\partial \theta}. \tag{134}$$

Hence the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in polar coordinates are

$$\frac{\partial}{\partial x} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r} \tag{135}$$

$$\frac{\partial}{\partial y} = \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial \theta} \tag{136}$$

Consequently

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\ \frac{\partial f}{\partial x}\quad \cos\theta\,\frac{\partial}{\partial r}\quad \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$-\frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\quad \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \tag{137}$$

$$= \quad \cos^2\theta\,\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\quad \frac{\partial^2 u}{\partial r\partial \theta} = \quad \frac{\sin^2\theta}{r^2}\,\frac{\partial^2 u}{\partial \theta^2} \quad + \quad \frac{\sin^2\theta}{r^2}\,\frac{\partial u}{\partial r} \quad + \quad \frac{2\sin\theta\cos\theta}{r^2}\,\frac{\partial u}{\partial \theta}. \tag{138}$$

Similarly

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \sin\frac{\partial}{\partial r}\ \sin\theta\frac{\partial u}{\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r}\ \frac{\partial}{\partial \theta}$$
$$\sin\theta\frac{\partial u}{\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \qquad (139)$$

$$= \sin^2\theta\,\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\ \frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\ \frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\ \frac{\partial u}{\partial r} -$$
$$\frac{2\sin\theta\cos\theta}{r^2}\ \frac{\partial u}{\partial\theta} \qquad (140)$$

Finally, adding (138) and (140), we have Laplace's equation in two dimensions

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0. \qquad (141)$$

SELF ASSESSMENT EXERCISES 3

Given that

$z = \sqrt{x^2 + y^2}$, $x = r\cos\theta$.

Find $\frac{\partial z}{\partial z}$ and show that $\frac{\partial z}{\partial 0}$ = 0.

3.10 Taylor's Theorem for Functions of Two Independent Variables

Theorem 2: (Taylor's theorem). If *f(x,y)* is defined in a region R of the xy-plane and ll its partial derivatives of orders up to and including the (n + 1)th are continuous in R, then for any point *(a,b)* in this region

$$f(a+h, b+k)=f(a,b)=*Df(a,b)=\frac{1}{2!}*D^2 f(a,b)+\ldots+\frac{1}{n!}*Df(a,b)+E_{n9} \qquad (142)$$

Where *D is the differential operator defined by (92) – (97) as

$$*D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}, \qquad (143)$$

and

$*D^r f(a,b)$ means $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial k}\right)^r f(x,y)$ (144)

evaluated at the point *(a,b)*. the Lagrange error term E_n is given by

$$E_n = \frac{1}{(n+1)!} *D^{n+1}f(a + \theta h, b + \theta k) \quad (145)$$

Where $0 < \theta < 1$.

To prove this theorem we let

$$x = a + ht, \quad y = b + kt, \quad (146)$$

where a, b, h k are constant and t is a variable. Then putting

$$f(x,y) = f(a + ht, \; b + kt) = u(t), \quad (147)$$

where *u(t)* is a continuous function of t, we have by (97)

$$\frac{d^n u}{dt^n} = *D^n f. \quad (148)$$

Since by assumption all partial derivatives of *f(x,y)* up to and including the (n + 1)th order are continuous in R so also are the ordinary derivatives of u with respect to t. Hence *u(t)* may be expanded by Maclaurin's series (see Chapter 6, 6.1 (6) and (7)) as

$$u(t) = u(0) + ty'(0) + \frac{t^n}{n!}u^{(n)}(0) + E_{n9} \quad (149)$$

where

$$E_n = \frac{t^{n+1}}{(n+1)!} u^{(n+1)}(\theta t), \quad 0 < \theta < 1. \quad (150)$$

Hence using (146) and (147) we have

$$f(a + ht, \; b + kt) = f(a,b) + t{*D}f(a,b) + \frac{t^2}{2!}{*D}f(a,b) + \ldots.. + \frac{t^n}{n!} {*D}f(a,b) + E_{n9} \quad (151)$$

where now

$$E_n = \frac{t^{n+1}}{(n+1)!} *D^{n+1}f(a + h\theta t, \; b + k\theta t), \; 0 < \theta < 1. \quad (152)$$

Putting t = 1 in (151) and (152) we finally obtain Taylor's expansion (142) with the error term (145).

Theorem 3: If

$$\lim_{n\to\infty} E_n = 0, \tag{153}$$

then

$$f(a + ht,\ b + kt) = f(a,b) + {}^*Df(a,b) + \frac{{}^*D^2}{n!} f(a,b) + \ldots$$
$$= \sum_{r=0}^{\infty} \frac{1}{r!} {}^*D^r f(a,b). \tag{154}$$

In all that follows we shall assume that (153) is satisfied.

An alternative form of Taylor's series (154) may be obtained by putting

$$h = x - a, \quad k = y - b. \tag{155}$$

Then

$$f(x,y) = f(a,b) + [(x - a)f_x(a,b) + (y - b)\, f_y(a,b)]$$
$$+ \frac{1}{2!}\{(x - a)^2 f_{xy}(a,b) + 2(x - a)(y - b)f_{xy}(a,b)$$
$$+ (y - b)^2 f_{yy}(a,b)\} + \ldots, \tag{156}$$

which is Taylor's expansion of $f(x,y)$ about the point (a ,b). When there is no dependence on y, (156) reduces to Taylor's series for a function of one variable (6.1 (8)).

Example 16: Expand the function

$$f(x,y) = \sin xy \tag{157}$$

about the point $1, \frac{\pi}{3}$ neglecting terms of degree three and higher

Here

$$f\ 1, \frac{\pi}{3} = \frac{\sqrt{3}}{2},$$

$$f_x(x,y) = y \cos xy,\ f_x\ 1, \frac{\pi}{3} = \frac{\pi}{6},$$

$f_y(x,y)$ = x cos xy, $f_y\left(1,\frac{\pi}{3}\right) = \frac{1}{2}$,

$f_{xy}(x,y)$ = -y^2 sin xy, $f_{xy}\left(1,\frac{\pi}{3}\right)$ = - $\frac{\pi^2\sqrt{3}}{18}$, (158)

$f_{xy}(x,y)$ = -xy sin xy = cos xy, $f_{xy}\left(1,\frac{\pi}{3}\right)$ = - $\frac{\pi^2\sqrt{3}}{6} + \frac{1}{2}$,

$f_{yy}(x,y)$ = -x^2 sin xy, $fy_{xy}\left(1,\frac{\pi}{3}\right)$ = - $\frac{\sqrt{3}}{2}$.

Hence substituting these results in (156) we have

$$\text{Sin xy} = \frac{\sqrt{3}}{2} + \left[(x-1)\frac{\pi}{6} + \left(y-\frac{\pi}{3}\right)\frac{1}{2}\right] + \frac{1}{2!}\left\{(x-1)^2\left(-\frac{\pi^2\sqrt{3}}{18}\right)\right.$$
$$\left. +2(x-1)\left(y-\frac{\pi}{3}\right)\left(-\frac{\pi\sqrt{3}}{6}+\frac{1}{2}\right) + \left(y-\frac{\pi}{3}\right)^2\left(-\frac{\sqrt{3}}{2}\right)\right\} \quad (159)$$

+ terms of degree 3 and higher

3.11 Maxima and Minima of Function of Two Variables

A function *f(x,y)* is said to have a maximum value at a point *(x,y)* = (a, b) if

f(a + h, b +k) – *f(a,b)* < 0, (160)

where h and k are small arbitrary quantities.

Similarly *f(x,y)* is said to have a minimum at *(x,y) = (a,b)* if

f(a + h, b +k) – *f(a,b)* > 0, (161)

These results may be interpreted geometrically (see Fig. 9.3) by noticing (in the manner of 9.1) that the surface z = *f(x,y)* is higher

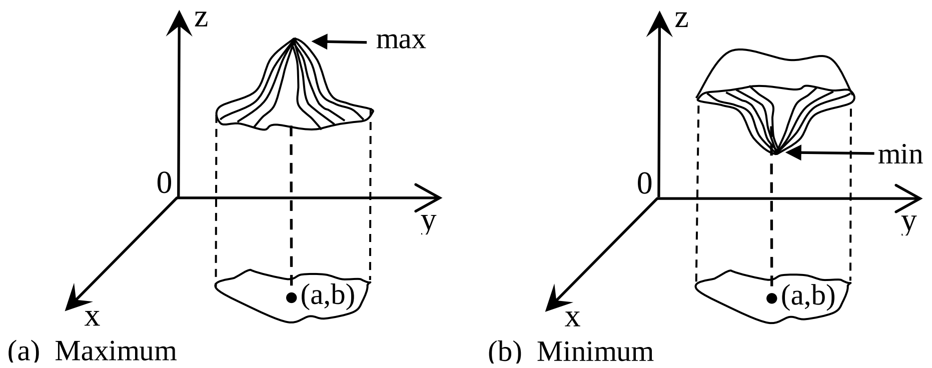


Fig. 2.3

at *(x,y) = (a,b)* than at any neighbouring point when (160) is satisfied (thus corresponding to a maximum), and is lower at a *(a,b)* than at any neighbouring point when (161) is satisfied (thus corresponding to a minimum).
Now if a maximum or minimum occurs at *(a,b)* the curves lying in the two planes x = a and y = b must also have maxima and minima or minima at *(a,b)* (see Fig. 9.4). Consequently the tangents T_1 and T_2 to these curves at *(a,b)* must be parallel to the Ox and Oy axes respectively.

This requires

$$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0 \qquad (162)$$

at all maxima and minima. The solution of these equations gives the coordinates of points of possible maxima and minima, and also of points called saddle points which will be defined later. In general we speak of the solution of (162) as giving the stationary or critical

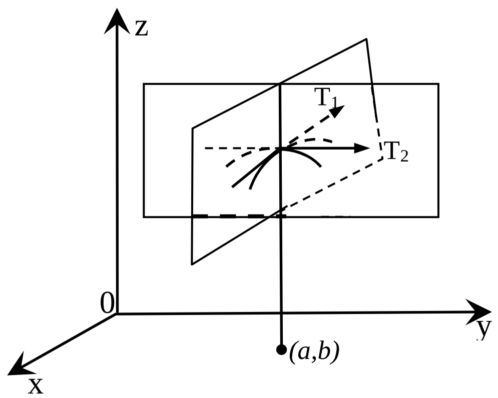


Fig. 2.4

Points of *f(x,y)*. To decide whether a particular stationary point is a maximum, minimum or neither we now use the Taylor expansion of fx, y) in the form given by (142), namely

$$f(a + h, b + k) = f(a,b) + *Df(a,b) + \frac{1}{2!}*Df(a,b) + \ldots, \qquad (163)$$

Where *D = $h\frac{\partial}{\partial x} + k\frac{\partial}{\partial k}$.

If *(a,b)* is a stationary point then (162) gives

*D$f(a,b)$ = 0. (164)

Hence, neglecting terms of order h^3, k^3 and higher, we have

$f(a + h, b + k) - f(a,b) = \frac{1}{2}\{h^2 f_{xy}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\}$, (165)

at a stationary point where, for example, $f_{xy}(a,b)$ means $\frac{\partial^2 f(x,y)}{\partial x^2}$ evaluate at *(a,b)*. We now see that (165) may be rewritten as either

$f(a + h, b + k) - f(a,b) = \frac{1}{2f_{xx}(a,b)}\{[h f_{xy}(a,b)]^2 - k^2[f_{xy}^2(a,b) - f_{xy}(a,b) f_{yy}(a,b)]\}$, (166)

or

$f(a + h, b + k) - f(a,b) = \frac{1}{2f_{yy}(a,b)}\{[h f_{xy}(a,b) + k f_{yy}(a,b)]^2 - h^2[f_{xy}^2(a,b) - f_{xy}(a,b) f_{yy}(a,b)]\}$, (167)

Clearly the sign of $f(a + h, b + k) - f(a,b)$, which by (160) and (161) is crucial in deciding whether a particular stationary point is a maximum or minimum, is now, by (166) and (167), dependent on the values of h and k. However, if

$\Delta \equiv f_{xy}^2(a,b) - f_{xy}(a,b) f_{yy}(a,b) < 0$ (168)

then the terms in curly brackets in (166) and (167) are positive for all h and k.

Consequently, with $\Delta < 0$, the sign of $f(a + h, b + k) - f(a,b)$ depends entirely on the signs of $f_{xy}(a,b)$ and $f_{yy}(a,b)$. From (160) and (161) we deduce therefore that *(a,b)* is a maximum if

$\Delta < 0$, $f_{xy}(a,b) < 0$, (169)

and a minimum if

$\Delta < 0$, $f_{xy}(a,b) > 0$, (170)

We note that $\Delta < 0$, and $f_{xy}(a,b) \gtrless 0$ simply $f_{xy}(a,b) \gtrless 0$.

When $\Delta > 0$ the signs of the curly brackets in (166) and (167) depend on the values of h and k. In this case the stationary point (a, b0 is called a saddle point. Such a point is neither a maximum nor a minimum, but is such that the point P is a maximum for the curve C_1 and a minimum for the curve C_2 (see fig. 9.5).

When $\Delta = 0$ a more refined test is required to determine the nature of a given stationary point.

Example 17: Consider the function

$$f(x,y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1. \tag{171}$$

The conditions $\frac{\partial f}{\partial x}x = 0$, $\frac{\partial f}{\partial y} = 0$ give the two equations

$$4x(x^2 + 2y^2 - 1) = 0, \tag{172}$$

and

$$4y(1 + 2x^2) = 0, \tag{173}$$

respectively.

Hence solving (172) and (173) we have

$$x = 0, \pm 1,$$

$$y = 0, \tag{174}$$

giving the stationary points of (171) as (0, 0), (1, 0) and (-1, 0). We now test each f these points separately for a maximum, minimum or saddle point. To do this we first differentiate (171) twice to get

$$\left.\begin{aligned} f_{xy} &= 12x^2 + 8y^2 - 4, \\ f_{yy} &= 8x^2 + 4, \\ f_{xy} &= 16xy. \end{aligned}\right\} \tag{175}$$

Point (0, 0). Using (175) we now have

$$f_{xy}(0, 0) = -4,\ f_{yy}(0, 0) = 4,\ f_{xy}(0, 0) = 0, \tag{176}$$

whence

$$\Delta = f^2_{xy}(0, 0) - f_{xy}(0, 0)f_{yy}(0, 0) = 16 > 0. \qquad (177)$$

This point is therefore a saddle point.
Point (1, 0). Here

$$f_{xy}(1, 0) = 8, \ f_{yy}(1, 0) = 12, \ f_{xy}(1, 0) = 0 \qquad (178)$$

and

$$\Delta = f^2_{xy}(1, 0) - f_{xy}(1, 0)f_{yy}(1, 0) = -96 < 0. \qquad (179)$$

By (170) this point is therefore a minimum.
Point (-1, 0). Since the values of $f_{xy}(-1, 0)$, $f_{yy}(-1, 0)$ and $f_{xy}(-1, 0)$ are identical with those given in (178), this point is also a minimum.

The function $f(x,y)$ defined in (171) therefore has two minima (at (1, 0), (-1, 0)), and one saddle point (at (0, 0)). Finally it is easily found that $f(x,y) = -2$ at both minima, and $f(x,y) = -1$ at the saddle point. The reader should now attempt o sketch the surface $z = f(x,y)$ defined by (171).

Example 18: To find the maximum value of

$$f(x, y, z) = x^2y^2z^2 \qquad (180)$$

subject to the condition

$$x^2 + y^2 + z^2 = c^2, \qquad (181)$$

where c is a constant. Problems of this type where some constraint is applied (which effectively means that not all the variables are independent) are best dealt with by the method of Lagrange multipliers (see next section). However, in this particular example we can easily reduce the problem to one in two-independent variables by eliminating z to get

$$f(x,y) = x^2y^2(c^2 - x^2 - y^2) \qquad (182)$$

and proceeding as in Example 17.
The stationary points are easily found to be (0, 0, c) and $\left(\pm\frac{c}{\sqrt{3}}, \pm\frac{c}{\sqrt{3}}, \pm\frac{c}{\sqrt{3}}\right)$, where all possible combinations of sign are allowed. This second point is, in fact, eight symmetrically placed points in the form of a cube centred at the origin of the 0xyz- coordinate

system. It is at these points that the function (180) takes on its maximum value $\left(=\frac{c^6}{27}\right)$.

3.12 Lagrange Multipliers

In the last example, a problem of maximising a function of three independent variables subject to a constraint was successfully dealt with by eliminating one of the variables. However, this approach may not always be possible. For example, if instead of the constraint (181), we had the relation $e^{-yz}\sin^2(x + z)+1 = 0$, it would not be possible to solve explicitly for z. Lagrange developed an alternative method of dealing with maxima and minima problems which are subject to constraint and which overcomes this difficulty. We indicate this method here for the case of functions of two variables only, but the technique maybe extended to any number of variables.

Suppose *f(x,y)* is to be examined for stationary points subject to the constraint

$$g(x, y) = 0. \qquad (183)$$

Now for *f(x, y)* to be stationary we must have the total differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \qquad (184)$$

This would normally lead to the usual equations

$$\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0 \qquad (185)$$

for the stationary points. However, *dx* and *dy* are not now independent but are related via the total differential of *g(x,y)*

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 \qquad (186)$$

(using (183)).

Hence multiplying (186) by a parameter λ and adding to (184), we have

$$d(f + \lambda g = \left(\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}\right)dy = 0. \qquad (187)$$

We now choose λ such that

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0, \quad (188)$$

whence if follows from (187) that

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial x} = 0. \quad (189)$$

Equations (188), (189) and (183) are together sufficient to determine the stationary points and the value of the multiplier λ.

Example 19 To find the maximum distance from the origin (0, 0) to the curve

$$3x^2 + 3y^2 + 4xy - 2 = 0. \quad (190)$$

Here we have to find the maximum value of the distance l, where

$$l^2 = f(x,y) = x^2 + y^2 \quad (191)$$

subject to the constraint

$$g(x,y) = 3x^2 + 3y^2 + 4xy - 2 = 0. \quad (192)$$

Now the Lagrange equations (1880 and (189) give

$$2x + \lambda(6x + 4y) = 0, \quad (193)$$

$$2y + \lambda(6x + 4y) = 0, \quad (194)$$

which must be solved together with (192). From (193) and (194) we find $4\lambda(y^2 - x^2) = 0$ whence $y = \pm x$. With $y = x$, (192) gives $10x^2 - 2 = 0$ of $x = \pm \frac{1}{\sqrt{5}}$; with $y = -x$, (192) gives $2x^2 - 2 = 0$ *or* $x = \pm 1$. The stationary points are therefore $\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, $\left(-\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$, (1, -1), (-1, 1).

From (191) we find that fro the first two points $l^2 = \frac{2}{5}$, whilst for the last two $l^2 = 2$. Hence the maximum distance from the origin to the curves is $l = \sqrt{2}$.

SELF ASSESSMENT EXERCISES 4

i. Prove that the volume of greatest rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ is } \frac{8abc}{3\sqrt{3}}.$$

ii. Find the stationary points of $f(x,y) = x^2 + y^2$ subject to the constraint $3x + 2y = 6$.

4.0 CONCLUSION

In this unit, we have considered and discussed into detail important mathematical concepts. The Partial Differentiation, we have also considered concepts like Lagrange Multiplier Techniques, which is a useful tool in determining the maxima and minima point in calculus of several variables.

The concepts developed will be useful in solving problems in more advanced mathematics as we progress in our studies.

5.0 SUMMARY

Here in this unit you have learnt about functions of several independent variables and various methods of performing derivatives on them.

You have also learnt about different type of functions that can be encountered in the process of performing the task of finding the derivatives of functions of several variables.

6.0 TUTOR-MARKED ASSIGNMENT

i. Find $\dfrac{\partial^3 f(x,y,z)}{\partial x \partial y \partial z}$ $f = (x,y,z)$ when

a) $f(x,y,z) = e^{xyz}$

b) $f(x,y,z) = \dfrac{xy}{2x+z}$ and verify in each case that $fxyz = fxyz = fxyz$.

ii. If $x = u^2 - v^2$, $y = u^2 + v^2$
show that

$$\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} = 8uv = \left\{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right\}^{-1}.$$

iii. Find the stationary points of the function

$$v = x^2 + y^2 + z^2$$

subject to the condition

$$x^2 - z^2 = 1.$$

iv. In determining the specific gravity by the formula $S = \frac{A}{A-W}$, where A is the weight in air, and W is weight in water. A can be read within 0.01gm and W can be read within 0.02gm. Find approximately the maximum error in S, if the readings are

A = 1.1gm and
W = 0.6gm. Find the maximum relative error $\frac{\Delta S}{S}$.

7.0 REFERENCES/FURTHER READING

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UNIT 3 CONVERGENCE OF INFINITE SERIES

CONTENTS

1.0 INTRODUCTION

In this unit we shall study convergence of series and sequences which is very useful in the subsequent development of this course.

2.0 OBJECTIVES

At the end of this study you should be able to do the following:

- test for convergence of series
- test for conditional convergence of series
- provide answers to the exercises at the end of this unit.

3.0 MAIN CONTENT

3.1 Definition

If a_1, a_2, a_3 … is a given sequence of numbers, the sum of the first n numbers is called the nth partial sum and is represented by

$$S_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{r=1}^{n} a_r. \qquad (1)$$

If the partial sums S_1, S_2, S_3, … converge to a finite limit S, where

$$S = \lim_{n\to\infty} S_n, \tag{2}$$

Then S is defined as the sum of the infinite series

$$a_1 + a_2 + \ldots + = \sum_{r=1}^{\infty} a_r, \tag{3}$$

and the series is said to be convergent. When the sequence of partial sums tends to an infinite limit, or oscillates either finitely or infinitely, the series is said to be divergent.

Example 1: The series

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)} = \frac{1}{1.3} + \frac{1}{2.3} + \ldots, \tag{4}$$

has partial sums $S_1 = \frac{1}{2}$, $S_2 = \frac{2}{3}$, $S_3, = \frac{3}{4}$, $S_4 = \frac{4}{5}$, $S_5 = \frac{5}{6}$, … which with increasing n tend to unity. Hence the series is convergent with a sum S = 1. This result can also be obtained by using the method of differences to sum the finite series

$$S_n = \sum_{r=1}^{n} \frac{1}{r(r+1)} \tag{5}$$

and then letting $n \to \infty$ in the result. For writing the rth term as

$$a_r = \frac{1}{r} - \frac{1}{r+1}, \tag{6}$$

$$\left.\begin{aligned} \text{we have } a_n &= \frac{1}{n} - \frac{1}{n+1}, \\ a_{n-1} &= \frac{1}{n-1} - \frac{1}{n}, \\ &\vdots \qquad \vdots \qquad \vdots \\ a_2 &= \tfrac{1}{2} - \tfrac{1}{3}, \\ a_1 &= 1 - \tfrac{1}{2}, \end{aligned}\right\} \tag{7}$$

which, on adding, give

$$S_n = \sum_{r=1}^{n} a_r = \sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}. \tag{8}$$

Hence $S = \lim_{n\to\infty} S_n = 1$, as before.

Example 2: The geometric series

$$\sum_{r=1}^{\infty} ak^r = a(1 + k + k^2 + \dots) \qquad (9)$$

(where *a* is a constant) has an *n*th partial sum S_n given by

$$S_n = a\frac{1-k^n}{1-k}. \qquad (10)$$

Hence if $|k| < 1$,

$$S = \lim_{n\to\infty} S_n = \frac{a}{1-k} \qquad (11)$$

and series is convergent.

The series is divergent, however, when $|k| \geqq 1$, since the partial sum S_n either increases without limit as $n \to \infty$, or oscillates either finitely ($k = -1$) of infinitely ($k < -1$).

3.2 Theorems on Series

Theorem 1: The series $\sum_{r=1}^{\infty} a_r$ cannot converge unless $\lim_{n\to\infty} a_n = 0$. This may be proved by considering the (n -1)th and *n*th partial sums given by

$$S_{n-1} = a_1 + a_2 + \dots + a_{n-1}, \qquad (12)$$

and

$$S_n = a_1 + a_2 + \dots + a_n, \qquad (13)$$

Subtracting (12) from (13) we have

$$S_n - S_{n-1} = a_n. \qquad (14)$$

Now if the series converges to a sum S then

$$S = S_n = \lim_{n\to\infty} S_{n-1}, \qquad (15)$$

and hence, from (14) and (15),

$$\lim_{n\to\infty} a_n = 0. \tag{16}$$

This condition is necessary but not sufficient for convergence in that there are many series satisfying (16) which nevertheless do not converge. The harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + .. = \sum_{r=1}^{\infty} \frac{1}{r} \tag{17}$$

is a good example of this since, although

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0,$$
(18)

the sum of the series is infinite (see next section). However, the series $\sum_{r=1}^{\infty} \cos \frac{\pi r}{4}$, for example, cannot converge since $\lim_{n\to\infty} \cos \frac{\pi n}{4} \neq 0$.

Theorem 2: If $\sum_{r=1}^{\infty} a_r = S$, then $\sum_{r=1}^{\infty} ka_r = kS$, where k is a constant. This follows from the obvious identity

$$\sum_{r=1}^{n} ka_r = k \sum_{r=1}^{n} a_r \tag{19}$$

and proceeding to the limit $n \to \infty$.

Theorem 3: If $\sum_{r=1}^{\infty} a_r = S$ and $\sum_{r=1}^{\infty} b_r = T$, then

$$\sum_{r=1}^{\infty} (a_r + b_r) = S + T.$$

Again this theorem is proved by considering the identity

$$\sum_{r=1}^{n} (a_r + b_r) = \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} b_r \tag{20}$$

and letting $n \to \infty$.

Theorem 4: If $\sum_{r=1}^{\infty} a_r = S$, then $\sum_{r=1}^{\infty} a_r = S + a_0$, where a_0 is nay number.

Writing $\bar{S}_n = \sum_{r=1}^{n} a_r$ and $S_n \sum_{r=1}^{n} a_r$,

we have $\bar{S}_n = S_n + a_0$. (21)

Hence, letting $n \to \infty$ in (21), the theorem is proved. This theorem shows that nay new term may be introduced at the beginning of a series without affecting the convergence of the series. A simple extension of this result shows that the removal or insertion of a finite number of terms anywhere in the series does not affect its convergence.

SELF ASSESSMENT EXERCISES 1

Examine the following series for convergence.

i. $\sum_{r=1}^{\infty} \frac{1}{2r(r+1)}$

ii. $\sum_{r=1}^{\infty} \frac{r!}{10^r}$

iii. $\sum^{\infty} r \frac{r+1}{r^2+1}$

iv. $\sum \frac{r^r}{r^1}$

v. $\bar{Z} \frac{1}{\sqrt{(r^r + r}}$

3.3 Series of Positive Terms

When a series $\sum_{r=1}^{\infty} a_r$ consists only of positive terms ($a_r \geqq 0$ for all r) it must either converge, or diverge to + ∞; it clearly cannot oscillate. Numerous tests of convergence are known for series of this type, and four such tests are given below:

a) Comparison Test

If $\sum_{r=1}^{\infty} a_r$ is a series of positive terms, and if $\sum_{r=1}^{\infty} b_r$ is a series of positive terms that is known to converge, then $\sum_{r=1}^{\infty} a_r$ is convergent if $a_r \leqq b_r$ for all sufficiently large r. Similarly, If $\sum_{r=1}^{\infty} b_r$ is known to diverge then $\sum_{r=1}^{\infty} a_r$ is divergent if $a_r \geqq b_r$ for all sufficiently large r.

Since, by Theorem 4, the removal or insertion of a finite number of terms does not affect the convergence of a series, it can be assumed in the proof that follows that the condition $a_r \leqq b_r$ (and $a_r \geqq b_r$) holds for all r.

The proof of this test may easily be seen by a graphical argument. Suppose each term of a series represents the area of a rectangle of base equal to unity and height equal to the magnitude of the term (see Fig. 5.1). Then the sum of the series is represented by the sum of the

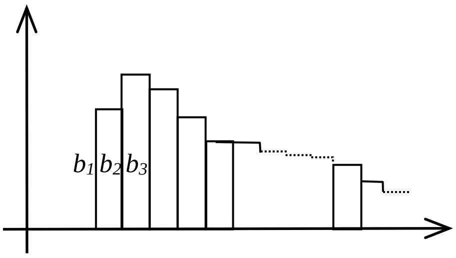

Fig. 3.1

areas of the rectangles. If now $\sum_{r=1}^{\infty} b_r$ converges to a sum then the total area of the rectangles must be finite, and if $a_r \leqq b_r$ for all r, the area of the rectangles representing the series a_r must also be finite. Hence $\sum_{r=1}^{\infty} a_r$ converges if $\sum_{r=1}^{\infty} b_r$ converges.

A similar argument applies to the second part of the test. An analytic proof of the test may be obtained by considering the partial sums

$$S_n = \sum_{r=1}^{\infty} a_r,\ T_r = \sum_{r=1}^{n} b_r. \tag{22}$$

Then $a_r \leqq b_r$ implies $S_n \leqq T_n$, and hence

$$\lim_{n\to\infty} S_n \leqq \lim_{n\to\infty} T_n = T, \tag{23}$$

where T is the sum of the convergent series $\sum_{r=1}^{\infty} b_r$. Now since $a_r \geqq$

0, S_n
never decreases and therefore
$$\lim_{n\to\infty} S_n = S \leqq T. \tag{24}$$

Hence, by (23), the first part of the test is proved. A similar argument exists for the second part of the test.

Example 3: The harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \sum_{r=1}^{\infty} \frac{1}{r} \tag{25}$$

may be shown to be divergent by writing it as

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots\right. \tag{26}$$

The terms in brackets are now greater than $\frac{1}{2}$; by grouping terms together in this way throughout the series so that the value of each group exceeds $\frac{1}{2}$ we see, by comparison with the divergent series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots, \tag{27}$$

that (26) is divergent. It should be noted, however, that bracketing of terms in series (as in (26)) is in general not possible without altering the character of the series (see 5.8).

Example 4: The p-series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots = \sum_{r=1}^{\infty} \frac{1}{r^p} \tag{28}$$

converges if $p > 1$, and diverges if $p \leqq 1$. We can prove these statements by taking the three cases $p \lesseqgtr 1$ separately:

a) if p =1, (28) becomes the harmonic series (25) and i consequently divergent;

b) if $p < 1$, each term of (28) (apart from the first) is greater than the corresponding term of the harmonic series. The series is therefore divergent; by comparison

c) if $p > 1$, we write the series as

$$1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots \qquad (29)$$

and continue grouping the terms throughout the series into brackets such that every brackets is less than the corresponding term of the series

$$1 + \frac{2}{2^p} + \frac{4}{4^p} + \ldots. \qquad (30)$$

Now (30) is a geometric series with a common ratio $k = 2^{1-p}$ which is known to be convergent for $|k| < 1$. Consequently (29) converges for $p > 1$.

b) Ratio Comparison Test

If $\sum\limits_{r=1}^{\infty} a_r$ and $\sum\limits_{r=1}^{\infty} b_r$ are two series of positive terms and

$$\frac{a_{r+1}}{a_r} \leqq \frac{b_{r+1}}{b_r}$$

for all sufficiently large r, then $\sum\limits_{r=1}^{\infty} a_r$ converge when $\sum\limits_{r=1}^{\infty} b_r$ converges. Similarly, if $\frac{a_{r+1}}{a_r} \geqq \frac{b_{r+1}}{b_r}$,

then $\sum\limits_{r=1}^{\infty} a_r$ diverges when $\sum\limits_{r=1}^{\infty} b_r$ diverges.

To prove these results we assume first that $\frac{a_{r+1}}{a_r} \leqq \frac{b_{r+1}}{b_r}$ for all r (see the remarks in (5.3 (a)). Then writing

$$a_r = \frac{a_r}{a_{r-1}} \cdot \frac{a_{r-1}}{a_{r-2}} \cdot \frac{a_{r-2}}{a_{r-3}} \cdots \frac{a_2}{a_1} . a_1 \leqq \frac{b_r}{b_{r-1}} \cdot \frac{b_{r-1}}{b_{r-2}} \cdot \frac{b_2}{b_1} . a_1 \qquad (31)$$

we have

$$a_r \leqq \frac{b_r a_1}{b_1}. \qquad (32)$$

Since (32) is true for all r, the comparison test shows that $\sum\limits_{r=1}^{\infty} a_r$ converges when $\sum\limits_{r=1}^{\infty} b_r$ converges. The second part of the test may be proved in the same way.

Example 5: The series $\sum_{r=1}^{\infty} a_r \equiv \sum_{r=1}^{\infty} r^{\frac{1}{3}}$ is convergent since, using the convergent series $\sum_{r=1}^{\infty} b_r \equiv \sum_{r=1}^{\infty} r^{\frac{1}{2}}$, we have

$$\frac{a_{r+1}}{a_r} = \left(\frac{r}{r+1}\right)^3 < \frac{b_{r+1}}{b_r} = \left(\frac{r}{r+1}\right)^2 \qquad (33)$$

for all r.

c) d'Alembert's Ratio Test

The series of positive terms $\sum_{r=1}^{\infty} a_r$ converges if $\lim_{r\to\infty} \frac{a_{r+1}}{a_r} = k < 1$, and diverges if $\lim_{r\to\infty} \frac{a_{r+1}}{a_r} = k > 1$. If $\lim_{r\to\infty} \frac{a_{r+1}}{a_r} = 1$, the series may either converge or diverge.

To prove the first part of this test we assume that

$$\lim_{r\to\infty} \frac{a_{r+1}}{a_r} = k < 1 \qquad (34)$$

and choose a number h such that $k < h < l$. Then for some sufficiently large value of r, say s, we have

$$\frac{a_{s+1}}{a_s} < h, \quad \frac{a_{s+2}}{a_{s+1}} < h, \quad \frac{a_{s+3}}{a_{s+2}} < h, \ \dots \qquad (35)$$

and so on.

Therefore

$$\left.\begin{aligned} &a_{s+1} < a_s h, \\ &a_{s+2} < a_{s\_1} h < a_s h^2, \\ &a_{s+3} < a_{s\_2} h < a_s h^3, \\ &\vdots \quad \vdots \quad \vdots \end{aligned}\right\} \qquad (36)$$

which give, on adding,

$$a_{s+1} + a_{s+2} + a_{s+3} + \dots < a_s(h + h^2 + h^3 + \dots). \qquad (37)$$

The series on the right-hand side of (37) is a convergent geometric series since, by assumption, $h < 1$. Hence, the series on the left-hand side (37) also converges. Finally therefore, if k < 1, $\sum_{r=1}^{\infty} a_r$ is convergent.

The case of k > 1 maybe proved in the same way. The ratio test clearly gives no information when k = 1 as can be seen by considering the p-series for which

$$\lim_{r\to\infty} \frac{a_{r+1}}{a_r} = \lim_{r\to\infty} \frac{r^p}{(r+1)^p} = \lim_{r\to\infty} \left(1+\frac{1}{r}\right)^{-p} = 1 \qquad (38)$$

for all p.

Example 6: The series

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} = \ldots = \sum_{r=1}^{\infty} \frac{r}{2^r} \qquad (39)$$

converges since

$$\lim_{r\to\infty} \frac{a_{r+1}}{a_r} = \lim_{r\to\infty} \left\{ \frac{r+1}{r} \left(\frac{2^r}{2^{r+1}} \right) \right\} = \frac{1}{2} \lim_{r\to\infty} \left(1+\frac{1}{r}\right) = \frac{1}{2}. \qquad (40)$$

d) Cauchy's Integral Test

If $\sum_{r=1}^{\infty} a_r$ is a series of positive decreasing terms and if there exists, for $x \geq 1$, a positive, monotonic decreasing integrable function f(x) such that $f(r) = a_r$ for r = 1, 2, 3 … n, then

$$0 < \sum_{r=1}^{n} a_r - \int_1^{n+1} f(x)\,dx < f(1). \qquad (41)$$

It may be further proved that

$$\lim_{n\to\infty} \left(S_n - \int_1^{n+1} f(x)\,dx \right) \qquad (42)$$

is finite. A direct consequence of (42) is that the series $\sum_{r=1}^{\infty} a_r$ converges when $\int_1^{\infty} f(x)dx$ converges (in the sense of Chapter 4, 4.8), and diverges when $\int_1^{\infty} f(x)dx$ diverges.

A simple proof of (41) maybe easily obtained using the type of graphical argument given in proving the comparison test.

Consider first the area *ABCD* shown in Fig. 5.2. Then, since *AB* = a_1 and *AD* = 1, we have that

area ABCD = a_1

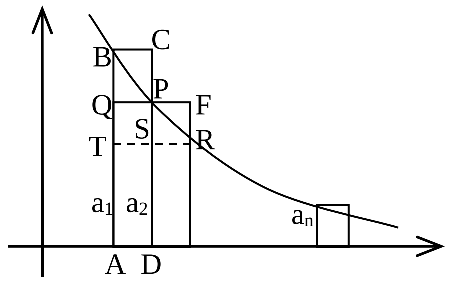

Fig. 5.2

The area under curve f(x) between A and D is $\int_1^2 f(x)dx$

Consequently

$$a_1 - \int_1^2 f(x)dx = \text{area BCP} < \text{area BCPQ}. \qquad (43)$$

Similarly, considering the next rectangle of height a_2

$$a_2 - \int_1^2 f(x)dx = \text{area PFR} < \text{area PFRS} = \text{area QPST}. \qquad (44)$$

After n such expressions we have, on adding,

$$0 < (a_1 + a_2 + a_3 + \dots + a_n) - \int_1^{n+1} f(x)dx < \text{area ABCD} = f(1),$$

(45) which proves the basic inequality (41) of the integral test.

Example 7: The series $\sum_{r=2}^{\infty} \frac{1}{r(\log r)^p}$ converges only if p > 1. This follows from the integral test since

$$I = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\log x)^p} = \lim_{b\to\infty} \int_{\log_e 2}^{\log_e b} \frac{du}{u^p} \qquad (46)$$

Converges only if p > 1.

Example 8: Using the divergent harmonic series in the integral test we have (from (41))

$$0 < 1 + 2 + \frac{1}{3} + \dots + \frac{1}{n} - \log_e(n + 1) < 1. \qquad (47)$$

Furthermore by (42)

$$\lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log_e(n+1) = \gamma, \qquad (48)$$

where $0 < \gamma < 1$. The constant γ is called Euler's constant and is approximately equal to 0.5772.

SELF ASSESSMENT EXERCISE 2

Examine the following series for convergence

i. $\sum_{r=1}^{\infty} \frac{1}{2r(r+1)}$

ii. $\sum_{r=1}^{\infty} \frac{r!}{10r}$

iii. $\sum_{r=1}^{\infty} r^2 x^r (x > 0)$

3.4 Alternating Series

If $\sum_{r=1}^{\infty} a_r$ is a series of term which are alternately positive and negative, and if the terms continually decrease in magnitude and $\lim_{n\to\infty} a_n = 0$, then the series converges.

Suppose

$$\sum_{r=1}^{\infty} a_r = a_1 - a_2 + a_3 - a_4 + a_5 \ldots , \qquad (49)$$

where $a_1\ a_2\ a_3 \ldots$ are positive decreasing terms. Plotting the values of the first few partial sums S_1, S_2, $S_3 \ldots$ along the line Ox (see Fig. 5.3) it is clear that these partial sums approach more and more closely to a definite value S. Hence, the series converges.

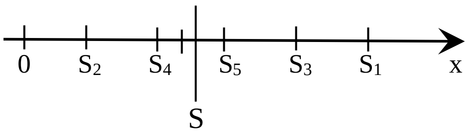

Fig. 5.3

Example 9: The series

$$1- \frac{1}{2}+\frac{1}{3}-\frac{1}{4} \ldots . = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \qquad (50)$$

satisfies all the conditions stated above and therefore converges. The sum of this series is $\log_e 2$.

3.5 Absolute Convergence and Conditional Convergence

Suppose

$$\sum_{r=1}^{\infty} a_r = a_1 + a_2 + a_3 + \ldots \qquad (51)$$

is a sries of positive and negative terms. Then

$$\sum_{r=1}^{\infty} |a_r| = |a_1| + |a_2| + |a_3| = \ldots \qquad (52)$$

is a series of positive terms which are just the absolute values of a_r. If (52) is convergent, (51) is said to be absolutely convergent, and it can be proved that any absolutely convergent series is also convergent. If, however, $\sum_{r=1}^{\infty} |a_r|$ diverges, but $\sum_{r=1}^{\infty} a_r$ converges, then $\sum_{r=1}^{\infty} a_r$ is said to be conditionally convergent. For example, the series

$$1- \frac{1}{2}+\frac{1}{3}-\frac{1}{4}+. = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \qquad (53)$$

discussed in Example 9 is conditionally convergent since the series formed from the absolute values of its terms

$$1+ \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots = \sum_{r=1}^{\infty} \left|\frac{(-1)^{r+1}}{r}\right| \qquad (54)$$

is the divergent harmonic series.

On the other hand, the series

$$1- \frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\ldots = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^2} \qquad (55)$$

is absolutely convergent since

$$1+ \frac{1}{2^2}+\frac{1}{3^2}+\ldots = \sum_{r=1}^{\infty} \left|\frac{(-1)^{r+1}}{r^2}\right| \qquad (56)$$

is convergent (p-series with p = 2).

SELF ASSESSMENT EXERCISES 3

i. Examine the following for convergence:

$$\sum_{r=1}^{\infty} \frac{\cos r\partial}{R} \text{ for } \partial = 0, \ \pi/2 \text{ and } 2\pi/3 .$$

ii. Show that if the conditionally convergence series
$1- \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+ \ \ldots$ is rearranged as the series
$\left(1+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}-1/4\right)+$
where two positive terms always alternate, with one negative term the series diverges.

3.6 Absolute Convergence Tests

Since $\sum_{r=1}^{\infty} |a_r|$ is series of positive terms its convergence may be discussed using any of the tests given in 5.3. For example, d'Alambert's ratio test for absolute convergence now takes the form: the series of positive and negative terms $\sum_{R=1}^{\infty} a_r$ is absolutely convergent (and hence convergent) if

$$\lim_{r\to\infty} \left|\frac{a_{r+1}}{A_r}\right| = k < 1, \tag{57}$$

and is divergent if

$$\lim_{r\to\infty} \left|\frac{a_{r+1}}{A_r}\right| = k > 1. \tag{58}$$

As before, the test does not decide between absolute convergence and divergence when k = 1.

Example 10: If

$$\sum_{r=1}^{\infty} a_r = 1 + 2x + 3x^2 + \ldots, \tag{59}$$

then

$$\lim_{r\to\infty} \left|\frac{a_{r+1}}{A_r}\right| = \lim_{r\to\infty} \left|\frac{(r+1)x^r}{Rx^{r-1}}\right| = |x| \lim_{r\to\infty} \frac{r+1}{R} = |x|. \tag{60}$$

Hence, when $|x| < 1$, (59) is absolutely convergent, and when $|x| > 1$, it is divergent. The question of what happens when $|x| = 1$ may be answered by taking the two possible cases $x = +1$ and $x = -1$ separately. When $x = +1$, (59) becomes

$$1 + 2 + 3 + 4 + \ldots, \tag{61}$$

which is divergent; similarly when $x = -1$ the series becomes

$$1 + 2 + 3 - 4 \ldots, \tag{62}$$

which is divergent since $\lim_{r\to\infty} a_r \neq 0$.

Hence (59) is absolutely convergent for $|x| \geqq 1$.

Example 11: The series

$$\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \ldots = \sum_{r=1}^{\infty} \frac{\sin rx}{r^2} \tag{63}$$

is absolutely convergent for all x, since, using the comparison test,

$$\left|\frac{\sin rx}{r^2}\right| \leqq \frac{1}{r^2} \tag{64}$$

for all r, and $\sum_{r=1}^{\infty} \frac{1}{r^2}$ is known to converge.

3.7 The Product of Two Series

If $\sum_{r=1}^{\infty} a_r$ and $\sum_{r=1}^{\infty} b_r$ are two absolutely convergent series, the $\sum_{r=1}^{\infty} c_r$, where

$$c_r = a_1 b_r + a_2 b_{r-1} + \ldots + a_r b_1 \tag{65}$$

is called the Cauchy product of $\sum_{r=1}^{\infty} a_r$ and $\sum_{r=1}^{\infty} b_r$, and is itself absolutely convergent. Furthermore, if $\sum_{r=1}^{\infty} a_r$ converges to a sum S, and $\sum_{r=1}^{\infty} b_r$ converges to a sum T, then $\sum_{r=1}^{\infty} c_r$ converges to a sum ST. A similar result has already been given earlier for the sum (and difference) of two convergent series.

Example 12: The product of e^{2x} and e^{-x} may be written, using (65), as

$$e^{2x}e^{-x} = \left(1 + 2X + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots\right)\left(1 - x + \frac{x^2}{2!} - \ldots\right) \qquad (66)$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x. \qquad (67)$$

3.8 Rearrangement of Series

Any series formed from $\sum_{r=1}^{\infty} a_r$ by taking its terms in a different order is called a rearrangement of $\sum_{r=1}^{\infty} a_r$. For example,

$$1 + \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{4^2} \ldots \qquad (68)$$

is a rearrangement of the absolutely convergent series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \ldots \qquad (69)$$

such that two positive terms alternate with one negative term throughout the series. Similarly, two possible rearrangements of the conditionally convergent series

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots \qquad (70)$$

are $$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \ldots \qquad (71)$$

and $$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} + \frac{1}{7} \ldots \qquad (72)$$

In (71) two positive terms alternate with one negative term; both of these are different from the original series (70) in which one positive term alternates with one negative term. Since in any rearrangement of an infinite series the pattern of N positive terms alternating with M negative terms can be chosen at will and can be continued throughout the series, it would be surprising if the sum of a rearranged series were equal to the sum of the original series. It can be proved, however, that provided either we restrict ourselves to series of positive terms or to series that are convergent, the term may be rearranged in any way without affecting the sums of the series. This result is not true for series that are conditionally convergent, and any rearrangement of terms in a series of this type will usually lead to a series with a different sum. For example, (68) will have the same sum as (69), since (69) is convergent (p-series, p = 2). The series (70), however, is only conditionally convergent since the series formed from the absolute values of its terms is the divergent harmonic series. Hence, we must expect that the sums of the two rearranged series (71) and (72) will be different from each other and different from the sum of (70). By way of justifying this we now show how to find the sums of (70) and (71) so verifying that they are different.

Consider (70) first: then by 5.1 (2) we have the sum S defined by

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} \quad 1-\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n} \tag{73}$$

$$= \lim_{n\to\infty} \quad 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n} - 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} \quad . \tag{74}$$

Now from 5.3, Example 8, we have

$$\lim_{n\to\infty} \quad 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} -\log_e (n+1) \quad = \gamma ,$$

where γ is Euler's constant. Hence

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} -\log_e (n+1) = \varepsilon_n , \tag{75}$$

where $\varepsilon_n \to \gamma$ as $n \to \infty$.

$$S = \lim_{n\to\infty} [\{ \varepsilon_{2n} + \log_e(2n + 1)\} - \{ \varepsilon_n + \log_e(n + 1)\}] = \log_e 2, \tag{76}$$

since ε_{2n} and ε_n both tend to γ as $n \to \infty$.
Similarly the series (71) may be written as

$$S_{3n} = \quad 1+\frac{1}{3}-\frac{1}{2} \quad + \quad \frac{1}{5}+\frac{1}{7}-\frac{1}{4} \quad + \ldots + \quad \frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}$$

$$= \quad 1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{4n-1} \quad - \frac{1}{2} \quad 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{4n} - \frac{1}{2} \quad 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n} - 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} \tag{77}$$

Using (75), (77) becomes
$\{\varepsilon_{4n} + \log_e(4n + 1)\} - \frac{1}{2}\{\varepsilon_{2n} + \log_e(2n + 1)\} - \frac{1}{2}\{\varepsilon_n + \log_e(n + 1)\}$,

which tends to $\frac{3}{2}\log_e 2$ as $n \to \infty$, since ε_{4n}, ε_{2n} and ε_n all tend to γ as $n \to \infty$. Hence, the sum of (71) differs from the sum of (70) by a factor $\frac{3}{2}$.

3.9 Power Series

An important type of series is the power series defined by

$$\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 \ldots, \tag{78}$$

where a_0, a_1, a_2, … are constants. The value of x for which (78) converges may be found using d'Alembert's ratio test given in 5.6. Hence for the series to be absolutely convergent we must have (from (57))

$$\lim_{r\to\infty}\left|\frac{a_{r+1}x^{r+1}}{a_r x^r}\right| = |x| \lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| = k < 1. \tag{79}$$

This condition may be more conveniently expressed as

$$|x| < R, \tag{80}$$

where R, the radius of convergence, is defined by

$$R = \lim_{r\to\infty}\left|\frac{a_r}{a_{r+1}}\right|, \tag{81}$$

provided the limit exists.

Writing (80) in full as

$$-R < x < R, \tag{82}$$

We see that the series converges absolutely provided x lies in the open interval (see Chapter 1, 1.2) $-R$ to R. This interval is called the interval (or range) of convergence. When $k = 1$, the ratio test gives no information and consequently the series may converge or diverge at $|x| = R$ (that is, at $x = \pm R$). However, as we shall see in the following examples, we may test the convergence of the series for these two particular values of *x* by direct substitution into the series.

Finally, by the ratio test, a power series obviously diverges for any value of *x*, which lies outside the interval of convergence.

Example 13: The exponential series (see 6.2 (40))

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \tag{83}$$

is a absolutely convergent for all *x*, since by (79)

$$|x| \lim_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| = |x| \lim_{r\to\infty} \left|\frac{r!}{(r+1)!}\right| = 0 \tag{84}$$

irrespective of the value of *x*. Similarly, by (81), the radius of convergence is infinite and, by (82), the interval of convergence is therefore

$$-\infty < x < \infty. \tag{85}$$

Hence the series (83) represents the function e^x for all *x*.

Example 14: The series

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)} + \ldots \tag{86}$$

is convergent for $|x| < 1$, since by (81)

$$R = \lim_{r\to\infty} \left|\frac{a_r}{a_{r+1}}\right| = \lim_{r\to\infty} \left|\frac{2r+1}{2r-1}\right| = 1. \tag{87}$$

The interval of convergence is therefore

$$-1 < x < 1. \tag{88}$$

At the end points $x = \pm 1$, the series may converge or diverge. Putting $x = 1$ in (86), the series becomes

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots, \tag{89}$$

which converges by 5.4 since the terms alternate in sign and continually decrease in magnitude. At x = -1, the series behaves like the series

$$1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \ldots, \tag{90}$$

which diverges by the integral test.
Hence (86) converges if, and only if,

$$-1 < x \leqq 1. \tag{91}$$

3.10 Operation with Power Series

a) The sum difference or product of two power series with common intervals of convergence leads to a third series, which converges for the common interval of convergence of the first two series. (This result follows from the general properties of series given in 5.2 and 5.7)

b) The series obtained by term-by-term differentiation (or integration) of a given convergent power series is a power series with the same interval of convergence.

Consider the power series

$$S = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \ldots \tag{92}$$

which converges if

$$|x| < \lim_{r\to\infty} \left|\frac{a_r}{a_{r+1}}\right| = R. \tag{93}$$

Then

$$\frac{dS}{dx} = \sum_{r=0}^{\infty} r a_r x^{r-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots \tag{94}$$

converges if

$$|x| < \lim_{r\to\infty} \left|\frac{r a_r}{(r+1)a_{r+1}}\right| = \lim_{r\to\infty} \left|\frac{r}{r+1}\right| + \lim_{r\to\infty} \left|\frac{a_r}{a_{r+1}}\right| = R \tag{95}$$

Similarly,

$$\int S\ dx = \sum_{r=0}^{\infty} \frac{a_r x^{r+1}}{r+1} = a_0 x + \frac{a_1 x^2}{2} + \frac{a_1 x^3}{3} + \ldots \tag{96}$$

converges if

$$|x| < \lim_{r\to\infty}\left|\frac{(r+2)a_r}{(r+1)a_{r+1}}\right| = \lim_{r\to\infty}\left|\frac{r+2}{r+1}\right| \lim_{r\to\infty}\left|\frac{a_r}{a_{r+1}}\right| = R. \qquad (97)$$

Hence, the differentiated and integrated series have the same intervals of convergence as the series from which they are derived. However, it does not follow that if the series converges at one (or both) of the end points ($x = \pm R$) of the interval of convergence that the differentiated or integrated series necessarily also converges at these points. As before, the convergence of these series at the two ends must be considered separately. Furthermore, we may prove that differentiating or integrating a power series term-by-term within its interval of convergence is the same as differentiating or integrating the function it represents.

c) If two power series converge for a common interval of convergence then one series may be substituted into the other to give a third series which converges in that common interval. For example, the series for $e^{e^{-x}}$ may be obtained by writing $y = e^{-x}$ and using the series

$$e^y = 1 + y + \frac{y^2}{2!} + 3 + \ldots \qquad (98)$$

Hence
$e^{-x}=$

$$1+\left(1-x+\frac{x^2}{2!}...\right)+\frac{1}{2!}\left(1-x+\frac{x^2}{2!}-...\right)^2+\frac{1}{3!}\left(1-x+\frac{x^2}{2!}-...\right)^3+\ldots(99)$$

4.0 CONCLUSION

In this unit, we have considered series and convergence of series. We have examined the condition under which a given series will converge conditionally. We also studied differentiation and integration of series and pointed out that this will always possible for infinite series of arbitrary functions, example of the case in point was considered in the unit.

5.0 SUMMARY

This unit is on the convergence of infinite series. It has a lot of application in higher mathematics. The unit will be of immense importance in the subsequent course in mathematical analysis.

6.0 TUTOR-MARKED ASSIGNMENT

i. Prove that the binomial series

$1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3\ldots+$converges if $-l < x < 1$

ii. Find the values of x for which the series

$1 - 2(x - 1) + 3(x - 1)^2 + \ldots + r(-1)^{r-1}(x - 1)^{r-1} + \ldots$ converges

iii. Show by integrating the series

$$S = \sum_{r=1}^{\infty} rx^{r-1}$$

term-by-term and summing and then differentiate the sum that $S = (1-x)^{-2}$ for what value **** is this valued.

7.0 REFERENCES/FURTHER READING

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UNIT 4 TAYLOR AND MACLAURIN SERIES

CONTENTS

1.0 INTRODUCTION

In this unit, we shall consider two special types of series expansion, namely Taylor and Maclaurin series.

Clearly, both Taylor series and Maclaurin series only represent the function *f(x)* in their interval of convergence.

When functions are expanded at $x = a$ (say) we have Taylor's expansion and when functions are expanded or $x = 0$ then we have Maclaurin expansion.

We have devoted a whole unit to these important theorems because of their usefulness in the study of analytic functions, and calculus in general.

Ready carefully and pay attention to every details.

2.0 OBJECTIVES

After studying these units, you should be able to

- carry out expansion using Taylor's and Maclaurin methods.
- evaluate limit of the function given.
- apply the theorem to solution of some mathematical problems.

3.0 MAIN CONTENT

3.1 Taylor's Theorem

We now state an important theorem, which enables functions to be expanded in power series in *x* in a given interval. (Examples of the series representation of a few functions have already been given in Unit 3.

Theorem 1: (Taylor's Theorem). If *f(x)* is a continuous, single-valued function of *x* with continuous derivatives *f'(x)*, *f''(x)* … up to and including $f^{(n)}(x)$ in a given interval $a \leqq x \leqq b$, and if $f^{(n+1)-(x)}$ exists in $a < x < b$, then

f(x)=f(a)+

$$\frac{(x+a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+E_n(x) \text{ ,(1)}$$

where

$$E_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \qquad (2)$$

and $a < \xi < x$.

The term E_n is a remainder term and represents the error involved in approximating to *f(x)* by the polynomial

$$f(a)+\frac{(x-a)}{1!}f'(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a). \qquad (3)$$

An alternative form of (1) may be obtained by changing *x* to $a + x$.
Then

$$f(a+x)=f(a)+\frac{x}{1!}f'(a)+\frac{x^2}{2!}f''(a)+\ldots+\frac{x^2}{n}f^{(n)}(a)+E_n(x), \qquad (4)$$

Where now, from (2),

$$E_n(x)=\frac{x^{n+1}}{(n+1)}f^{(n+1)}(a+\theta x) \qquad (5)$$

and $0 < \theta < 1$.
A special case of (1) and (2) (or (4) and (5)) is when $a = 0$. Then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+E_n(x), \qquad (7)$$

and $0 < \theta < 1$.

Theorem 2: If $\lim_{n\to\infty} E_n(x) = 0$, then *f(x)* may be represented by the power series (see (1))

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots = \sum_{r=0}^{\infty} \frac{(x-a)^r}{r!} f^{(r)}(a), \quad (8)$$

Or its equivalent form (see (4))

$$f(a+x) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \ldots = \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(a). \quad (9)$$

These two series are Taylor's series for *f(x).*
The special case $a = 0$ gives, with $\lim_{n\to\infty} E_n(x) = 0$,

$$f(x) = f(0) + xf'(0) \frac{x^2}{2!} f''(0) + \ldots = \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0), \quad (10)$$

which is known a Maclaurin's series.

Clearly, both the Taylor series and the Maclaurin series only represent the function *f(x)* in their intervals of convergence. The Taylor series is often referred to as a series expansion of *f(x)* about the point $x = a$, and the Maclaurin series as an expansion about the point $x = 0$ (later we shall meet functions which have no Maclaurin series but which nevertheless can be expanded about some other point $x = a$, $a \neq 0$).

The form of Taylor's series maybe verified in the following way:

Let

$$f(x) = A_0 + A_1 + (x-a) + A_2(x-a)^2 + A_3(x-a)^3 + \ldots, \quad (11)$$

where $A_0, A_1, A_2 \ldots$ are constants. Then differentiating term-by-term we have

$$f'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \ldots \quad (12)$$

$$f''(x) = 2A_2 + 3.2A_3(x-a) + 4.3A_4(x-a)^2 + \ldots \quad (13)$$

$$f'''(x) = 3!A_3 + 4!A_4(x-a) + \ldots \quad (14)$$

and in general,

$$f^{(n)}(x) = n!A_n + (n+1)!A_{n+1}(x-a) + \ldots \quad (15)$$

Putting $x = a$ in (11)-(15) now gives

$$\left.\begin{array}{l} f(a)=A_0, f'(a)=A_1, \\ f''(a)=2!A_2, f'''(a)=3!A_3, \\ f^{(n)}(a)=n!A_n \end{array}\right\} \qquad (16)$$

Hence using these values for the constant $A_0, A_1, \ldots$in (11) we obtain the Taylor's series (8).

3.2 Standard Expansion

Before listing the Maclaurin series for some of the simple functions, we illustrate the use of the Taylor's and Maclaurin series by the following examples.

Example 1: Suppose we want to expand the function $f(x) = e^{3x}$ about $x = 0$ using Maclaurin's series.

Then since $f'(x)=3e^{3x}, f''(x)=9e^{3x}, \ldots f^{(n)}(x)=3^n e^{3x}$ and $f'(0)=3$, $f''(0)=9, \ldots f^{(n)}(0)=3^n$, we have from (6)

$$e^{3x}=1+3x+\frac{(3x)^2}{2!}+\frac{(3x)3}{3!}+\ldots+\frac{(3x)^n}{n!}+E_n(x), \qquad (17)$$

where, by (7)

$$E_n(x)=\frac{(3x)^{n+1}}{(n+1)!}e^{3\theta x}, (0<\theta<1. \qquad (18)$$

For any given finite value of x, say $x = c$, it is clear that

$$\lim_{n\to\infty} E_n(c)=0. \qquad (19)$$

Hence by (10), e^{3x} may be represented by the infinite series

$$1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\ldots=\sum_{r=0}^{\infty}\frac{(3x)^r}{r!}. \qquad (20)$$

Using the d'Alembert ratio test we find that (20) converges absolutely for all x. the possible error involved in approximating to e^{3x} (for a given value of x) by a finite number of terms of (20) may be found using (18) as follows.

Suppose $x = 0.02$ and $n = 3$. Then, since $(0 < \theta < 1$, E_n must satisfy the inequality relation

$$\frac{(0.06)^4}{4!} < E_n < \frac{(0.06)^4}{4!} e^{0.06}, \tag{21}$$

which gives (approximately)

$$5 \times 10^{-7} < E_n < 6 \times 10^{-7} \tag{22}$$

In other words, by taking only four terms (n = 3) of (20), the value of the resulting finite series for $x = 0.02$ differs from the exact value of $e^{3(0.02)}$ by a small number of the order of 5×10^{-7}.

On the other hand, with the same number of terms but with $x = \frac{1}{3}$, (18) gives

$$\frac{1}{4!} < E_n < \frac{e}{4!} \tag{23}$$

or $0.042 < E_n < 0.133$. (24)

Taking four terms of the Maclaurin series therefore is not a good approximation to e^{3x} for $x = \frac{1}{3}$ and more terms should be taken if the error is to be reduced.

Example 2: As an example of Taylor's expansion we expand the function $f(x) = \cos x$ about the point $x = \frac{\pi}{3}$. Differentiating we have $f'(x) = -\sin x$, $f''(x) = -\cos x$, and in general

$$f^{(n+1)}(x) = \cos\left(x + \frac{n+1}{2}\pi\right). \tag{25}$$

Hence

$$\left.\begin{aligned} f\left(\frac{\pi}{3}\right) &= \cos\frac{\pi}{3} = \frac{1}{2}, \\ f'\left(\frac{\pi}{3}\right) &= -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}, \\ f''\left(\frac{\pi}{3}\right) &= -\cos\frac{\pi}{3} = -\frac{1}{2}, \\ f'''\left(\frac{\pi}{3}\right) &= \sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}, \end{aligned}\right\} \tag{26}$$

and so on. Using these results in (1) we find

$$\cos x = \frac{1}{2} - \left(x - \frac{\pi}{3}\right)\frac{\sqrt{3}}{2} - \frac{\left(x - \frac{\pi}{3}\right)^2}{2!}\frac{1}{2} + \frac{\left(x - \frac{\pi}{3}\right)^3}{3!}\frac{\sqrt{3}}{2} + \ldots$$

$$= \frac{\left(x-\frac{\pi}{3}\right)^n}{n!} \cos\left(\frac{\pi}{3}+\frac{n}{2}\pi\right) + E_n(x), \qquad (27)$$

where, from (2) and (25),

$$E_n(x) = \frac{\left(x-\frac{\pi}{3}\right)^{n+1}}{(n+1)!}\cos\left(\xi+\frac{n+1}{2}\pi\right). \qquad (28)$$

Hence, since $\left|\cos\left(\xi+\frac{n+1}{2}\pi\right)\right| \le 1$, (28) may be written as

$$|E_n(x)| < \frac{\left|\left(x-\frac{\pi}{3}\right)^{n+1}\right|}{(n+1)!} \qquad (29)$$

Again (as in Example 1) for any given value of *x*, say x = c, $E_n(c)$ may be made as small as we please by choosing sufficiently large values of *n*. Hence, since $\lim_{n\to\infty} E_n = 0$, *cos x* may be represented by the infinite Taylor's series (8) as

$$\cos x = \frac{1}{2} - \left(x-\frac{\pi}{3}\right)\frac{\sqrt{3}}{2} - \left(x-\frac{\pi}{3}\right)^2\frac{1}{2.2!} + \left(x-\frac{\pi}{3}\right)^3\frac{\sqrt{3}}{2.3!}$$

$$+\ldots+\frac{\left(x-\frac{\pi}{3}\right)}{r!}\cos\left(\frac{\pi}{3}+\frac{r}{2}\right)+\ldots, \qquad (30)$$

which, by the ratio test converges for all *x*.

This series is useful in evaluating the cosines of angles without the use of tables. For example, $\cos 61^0$ may be evaluated by putting $\frac{61\pi}{180}$ radians in (30) which then gives

$$\cos 61^0 = \cos\frac{61\pi}{180} = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) - \frac{1}{2.2!}\left(\frac{\pi}{180}\right)^2 + \frac{\sqrt{3}}{2}.\frac{1}{3!}\left(\frac{\pi}{180}\right)^3 + \ldots. \qquad (31)$$

The error involved by taking a finite number of terms of this series may easily be estimated from (29). For example, with two terms (n =1)

$$\cos 61^0 \cong \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) = 0.4849 \qquad (32)$$

correct to four decimal places, with a possible error given by

$$\left|E_1\left(\frac{61\pi}{180}\right)\right| \leqq \frac{1}{2!}\left(\frac{\pi}{180}\right)^2 = 0.0001 \qquad (33)$$

to the same number of decimal places. The value of cos 61^0 obtained from tables is found to be 0.4848 (again corrected to four decimal places).

We now give the first few terms of the Maclaurin series for some elementary functions

i) $(1+x)^a = 1+\alpha x + \frac{\alpha\ \alpha-1}{2!}x^2 + \frac{\alpha\ \alpha-1\ \ \alpha-2}{3!}x^3 + \ldots$ for $|x| < 1$, where α is any real number, (34)

ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ for all x, (35)

iii) $\cos x = 1 - \frac{x^2}{2!} + \frac{2x^4}{4!} - \frac{6x^6}{6!} + \ldots$ for all x, (36)

iv) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, (37)

v) $\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $-1 < x \leqq 1$, (38)

vi) $\frac{1}{2}\log_e \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$ for $-1 < x < 1$, (39)

vii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ for all x, (40)

viii) $\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$ for all x, (41)

ix) $\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$ for all x. (42)

The series for functions, which are simple combination of these elementary functions, maybe obtained using the properties of power series treated in Unit 3. For example, substituting the series for *sin x* in the exponential series we have as in Unit 3, equation 10.

$$e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} + \ldots \qquad (43)$$

for all *x*.

Similarly (using 5.10 (a)).

$$\cosh x \sin x = x + \frac{x^3}{3} - \frac{x^5}{30} \ldots \qquad (44)$$

for all *x*.

Finally, it should be noted that functions like $\frac{e^{-x}}{x}$, $\log_e x$ and *cos x* have no Maclaurin expansions since they are not defined at *x = 0.*

Nevertheless, we may expand such functions about some other point using Taylor's series. For example, expanding $\log_e x$ about x = 1 we find

$$\log_e x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + ... \qquad (45)$$

for $0 < x \leqq 2$.

In deriving the Maclaurin series for certain functions it is sometimes convenient to use Leibnitz's formula given in Chapter 3, 3.6, to obtain the higher differential coefficients. We illustrate this method by an example.

Example 3: If y = sin ($m\ sin^{-1}x$), where m is a constant, then differentiating twice we find that y satisfies the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0. \qquad (46)$$

Using Leibnitz's formula, we have (for $n > 0$)

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (m^2 - n^2)\frac{d^ny}{dx^n} = 0, \qquad (47)$$

which gives, with $x = 0$,

$$\frac{d^{n+2}y}{dx^{n+2}} = (n^2 - m^2)\frac{d^ny}{dx^n} \qquad (48)$$

Hence (48) is a relation between the values of all the differential coefficients of y evaluated at $x = 0$; this is exactly what is required in developing the Maclaurin series for y. Since $y(0) = 0$, $y(0) = m$ it can be easily verified using (48) that

$$y = mx + \frac{m(1-m^2)x^3}{3!} + \frac{m(1-m^2)(9-m^2)x^5}{5!} + ... \qquad (49)$$

3.3 Evaluation of Limits

Suppose we have two functions $f(x)$ and $g(x)$ which are zero when x = a. Then although the ratio $\frac{f(a)}{g(a)}$ is an undefined quantity $\frac{0}{0}$, nevertheless the limit of $\left(\frac{f(x)}{g(x)}\right)$ as $x \to a$ may exist. An example of this type of ratio has already been met in Chapter 2, 2.4 where it was shown by a geometrical argument that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1.$ (50)

We now show how to proceed analytically with limits of this type. Consider the ratio of *f(x)* and *g(x)* and let both functions be expanded about the point *x* = *a* using Taylor's theorem. Then

$$\frac{f(x)}{g(x)} = \frac{f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots}{g(a)+(x-a)g'(a)+\frac{(x-a)^2}{2!}g''(a)+\ldots} \quad (51)$$

Now by assumption f(a) = g(a) = 0. Hence

$$\frac{f(x)}{g(x)} = \frac{f'(a)+\frac{(x-a)}{2!}f''(a)+\ldots}{g'(a)+\frac{(x-a)}{2!}g''(a)+\ldots} \quad (52)$$

and consequently

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}, \quad (53)$$

provided g'*(a)* is non-zero. Equation (53) states that the limit of the ratio of two functions as $x \to a$ where both functions are zero at *x* = *a* is given by the ratio of the derivatives of the functions each evaluated at *x* = *a*. If, however, *f'(a)* = *g'(a)* = 0 then the same procedure must be applied to the ratio $\frac{f'(x)}{g'(x)}$. Consequently if *f(a)* = *g(a)* = 0 and *f'(a)* = *g'(a)* = 0, we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f''(a)}{g''(a)}, \quad (54)$$

provided *g''(a)* is non-zero. Provided the limit exists it is usually possible to find a value of *n* such that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{n}(a)}{g^{n}(a)}. \quad (55)$$

This method of evaluating limits is sometimes more conveniently expressed by writing (53) as

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \quad (56)$$

which is usually known as l'Hospital's rule.
We illustrate these results by the following examples.

Example 4: Using (56)

$$\lim_{x\to 1}\left\{\frac{\log_e x}{x^2-1}\right\} = \lim_{x\to 1}\frac{1/x}{2x} = \frac{1}{2}. \qquad (57)$$

Example 5: To evaluate

$$\lim_{x\to 0}(\cos x)^{1/x} \qquad (58)$$

we put $y = (\cos x)^{1/x}$ and consider the behaviour of

$$\log_e y = \frac{\log_e \cos x}{x}. \qquad (59)$$

Then by (56)

$$\lim_{x\to 0}\log_e y = \lim_{x\to 0}\left\{\frac{\log_e \cos x}{x}\right\} = \lim_{x\to 0}\frac{-\tan x}{1} = 0. \qquad (60)$$

Hence, since as $x\to 0$, $\log_e y\to 0$, we have

$$y = (\cos x)^{1/x}\to 1. \qquad (61)$$

Example 6: This example illustrates the repeated use of l'Hospital's rule. For (by (56))

$$\lim_{x\to 0}\frac{\tan x - x}{x-\sin x} = \lim_{x\to 0}\left(\frac{\sec^2 x-1}{1-\cos x}\right). \qquad (62)$$

But $\frac{\sec^2 x-1}{1-\cos x}$ is of the form $\frac{0}{0}$ when $x = 0$. Hence, we apply l'Hospital's rule again which gives

$$\lim_{x\to 0}\left(\frac{\sec^2 x-1}{1-\cos x}\right) = \left(\frac{2\sec^2 x\tan x}{\sin x}\right) = \lim_{x\to 0}(2\sec^3 x) = 2. \qquad (63)$$

The second application of L'Hospital's rule could have been avoided by rewriting the right-hand side of (62) as

$$\lim_{x\to 0}\left(\frac{\sec^2 x - 1}{1-\cos x}\right) = \lim_{x\to 0}\left(\frac{1-\cos^{2x}\ \sec^2 x}{1-\cos x}\right) = \lim_{x\to 0}\{(1 + \cos x)\sec^2 x\} =$$

$$\lim_{x\to 0}\sec^2 x + \lim_{x\to 0}\sec x = 2, \qquad (64)$$

(using Theorem 1, Unit 1).

Example 7: If *f(x)* and *g(x)* both tend to infinity as $x \to a$, we may still apply l'Hospital's rule by writing

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\left\{\frac{1/g(x)}{1/f(x)}\right\}, \qquad (65)$$

where the ration $\left\{\frac{1/g(x)}{1/f(x)}\right\}$ is of the form $\frac{0}{0}$ at $x = a$.

Similarly if $\frac{f(x)}{g(x)}$ becomes either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x\to\infty$ we may write (putting $x = 1/y$)

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{y\to 0}\frac{f\left(\frac{1}{y}\right)}{g\left(\frac{1}{y}\right)} = \lim_{y\to 0}\left\{\frac{-\frac{1}{y^2}f'\left(\frac{1}{y}\right)}{-\frac{1}{y^2}g'\left(\frac{1}{y}\right)}\right\} = \lim_{y\to 0} \qquad (66)$$

Hence, l'Hospital's rule applies when $a \equiv \infty$.
For example,

$$\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty}\left\{\frac{x^3}{e^{x^2}}\right\} = \lim_{x\to\infty}\left\{\frac{3x^2}{2xe^{x^2}}\right\} = \frac{3}{2}\lim_{x\to\infty}\left\{\frac{x}{e^{x^2}}\right\}$$

$$= \frac{3}{2}\lim_{x\to\infty}\left\{\frac{1}{2xe^{x^2}}\right\} = 0 \qquad (67)$$

Similarly, if $n > 0$, it follows that

$$\lim_{x\to\infty}(x^n e^{-x^2}) = 0, \qquad (68)$$

and (by putting $x = 1/y$) that

$$\lim_{y\to 0}\left[\frac{e^{-1/y^2}}{y^n}\right] = 0. \qquad (69)$$

Example 8: The use of l'Hospital's rule may often be avoided by using series expansions. For example,

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left(\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-...}{x}\right) = \lim_{x\to 0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-...\right) = 1, \qquad (70)$$

as found earlier.

4.0 CONCLUSION

In this unit, you have studied Taylor and Maclaurin series expressions; you have studied the important theorem that enables us to carry out series expansion. We have also used the series expansion in the determination of limit of some functions.

5.0 SUMMARY

In this unit, you studied:

- Taylor and Maclaurin expansion
- Applied the technique to determine the limit of some difficult functions.
- That with clever application of Taylor's expansion, the use of l'Hospital's rules can be avoided in some functions.

6.0 TUTOR-MARKED ASSIGNMENT

i. If $y = e^{\sin^{-1}x}$ prove that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - y = 0$.
Hence verify the Maclaurin expansion
$e^{\sin^{-1}x} = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{5}{2_4}x^4\ldots$

ii. Prove that
$\sin x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!}(\cos\theta x)$
and that
$e^{-x^2} = 1 + x^2 + \dfrac{x^4}{2} + \dfrac{x^6}{6}\, e^{-0^2}\, x^2$

7.0 REFERENCES/FURTHER READINGS

G. Gstephenson(1977). Mathematical Methods for Science Students.

K. A Stroud and Dexter J.Booth(2001). Engineering Mathematics 5th Edition. Palgrave.

UNIT 5 NUMERICAL INTEGRATION

CONTENTS

1.0 INTRODUCTION

To evaluate an integral in terms of known functions is often impossible. For example the elliptic integral of the form $I = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^{2\theta}}}$ is not easy to evaluate directly without recourse to numerical integration.

In this unit, we shall apply the technique of trapezoidal rules and Simpson integration methods to solve numerically integral problems, we cannot evaluate quantitatively.

2.0 OBJECTIVES

By the end of this study, you should be able to perform numerical integration using the following techniques.

- trapezoidal rule
- simpson's rules.

3.0 MAIN CONTENT

3.1 Trapezium Rule

As mentioned in earlier units the evaluation of an integral in terms of known functions is often impossible. Furthermore, in some cases the integrand may only be defined by a set of tabulated values. To meet the difficulties, some numerical procedure is required, which will give a good approximation to the value of the integral. Clearly one of the simplest methods of doing this is to interpret the integral

$\int_a f(x)dx$ graphically as the area between the curve *y =f(x)*, the x-axis, and the lines *x* = *a*, *x* = *b*, and to estimate this area as accurately as possible. Consider, for example, the curve *y* = *f(x)* as shown in Fig. 3.1. Then to obtain an approximation

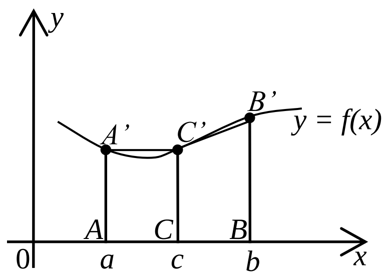

Fig. 3.1

to the required area we may draw in the straight lines *A'C* and *C'B* and evaluate the sum of the areas of the two trapeziums *ACC'A'* and *'CBB'C'*. If now the point *C* is chosen to be the mid-point of the range (*a, b*) such that *AC* = *CB* = *h*, then

area *ACC'A'* = $\frac{h}{2}(AA'+CC') = \frac{h}{2}\ f(a)+f\ \frac{a+b}{2}$ (1)

and

area *CBB'C'* = $\frac{h}{2}(CC'+BB') = \frac{h}{2}\ f\ \frac{a+b}{2}\ +f(b)$. (2)

Hence by adding (1) and (2) we have

$$\int_a^b f(x)dx \simeq \frac{h}{2}\ f(a)+2f\ \frac{a+b}{2}\ +f(b)\ . \quad (3)$$

This formula, usually known as the trapezium rule, gives a good approximation to the value of the integral when the curve *y* = *f(x)* deviate only slightly from the straight lines *A'C, C'B'*. When deviations occur, however, the accuracy may usually be improved by dividing the area under the curve into a larger (even) number of trapezium of smaller width and applying (3) to each pair. As an example of the trapezium rule, we now consider the numerical evaluation of a simple integral whose value is known exactly.

Example 1: If

$$I = \int_a^b f(x)dx = \int_1^3 \frac{dx}{x^2}, \quad (4)$$

then dividing the range of integral into two parts each of width h(= 1), we have by (3)

$$I \simeq \frac{1}{2}\{f(1)+2f(2)+f(3)\} = \frac{1}{2}\ 1+\frac{2}{4}+\frac{1}{9} = 0.81. \qquad (5)$$

This is to be compared with the exact value of 2/3. A better approximation may be obtained by dividing the area under the curve into four parts each of width $h(=\tfrac{1}{2})$ and applying (3) to each pair. In this way we find

$$I = \tfrac{1}{2}.\tfrac{1}{2}\{f(1)+2f(1.5)+f(2)\}+\tfrac{1}{2}.\tfrac{1}{2}\{f(2)+2f(2.5)+f(3)\} \qquad (6)$$

$$= \frac{1}{4}\ 1+\frac{2}{2.25}+\frac{1}{4}+\ \frac{1}{4}\ \frac{1}{4}+\frac{2}{6.25}+\frac{1}{9}\ , \qquad (7)$$

= 0.70.

3.2 Simpson's Rule

A better approximation to the area indicated in Fig. 3.1 maybe obtained in the following way. Suppose $x = c$ is the coordinate of the point C such that $a = c- h$, $b = c + h$. Then writing $x = c + y$, expanding by Taylor's series, and integrating term-by-term we obtain

$$\int_a^b f(x)dx = \int_{c-h}^{c+h} f(x)dx = \int_{-h}^{h} f(c+y)dy \qquad (8)$$

$$= \int_{-h}^{h}\left\{f(c)+yf^{(1)}(c)+\frac{y^2}{2!}f^{(2)}(c)+\ldots+\frac{y^r}{r!}f^{(r)}(c)+\ldots\right\}dy \qquad (9)$$

$$= 2h\left\{f(c)+\frac{h^2f^{(2)}(c)}{3!}+\frac{h^4f^{(4)}(c)}{5!}+\ldots+\frac{h^2f^{(2r)}(c)}{(2r+1)!}+\ldots\right\}, \qquad (10)$$

where, in general, $f^{(r)}(c)$ is the value of $\frac{d^r f}{dx^r}$ at $x = c$.

Now, since by Taylor's series

$$f(c+h) = f(c)+hf^{(1)}(c)+\frac{h^2}{2!}f^{(2)}(c)+\ldots+\frac{h^r}{r!}f^{(r)}(c)+\ldots \qquad (11)$$

$$f(c-h) = f(c)-hf^{(1)}(c)+\frac{h^2}{2!}f^{(2)}(c)+\ldots+\frac{(-1)^r h^r}{r!}f^{(r)}(c)+\ldots \qquad (12)$$

we also have

$f(c + h) = f(c - h) =$

$$2\left\{f(c)+\frac{h^2}{2!}f^{(2)}(c)+\frac{h^4}{4!}f^{(4)}(c)+\ldots+\frac{h^{2r}}{(2r)!}f^{(2r)}(c)+\ldots\right\} \qquad (13)$$

Hence neglecting terms involving h^4 and higher powers of h in (10) an (13), and eliminating $f^{(2)}(c)$, we finally obtain Simpson's formula

$$\int_a^b f(x)dx \simeq 2h\left\{f(c)+\frac{h^2}{3!}f^{(2)}(c)\right\} \qquad (14)$$

$$\simeq 2h\left\{f(c)+\frac{h^2}{3!}\left(\frac{f(c+h)+f(c-h)-2f(c)}{h^2}\right)\right\}, \qquad (15)$$

$$=\frac{h}{3}\{f(c-h)+4f(c)+f(c+h)\}, \qquad (16)$$

$$=\frac{h}{3}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]. \qquad (17)$$

The error involved here by approximating to the integral in this way is such that if

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]+E, \qquad (18)$$

then

$$E \simeq -\frac{h^4}{180}[f^{(3)}(b)-f^{(3)}(a)], \qquad (19)$$

where, as before, $f^{(3)}(b)$ and $f^{(3)}(a)$ mean the values of $\frac{d^3 f}{dx^3}$ at $x = b$ and $x = a$ respectively.

As with the trapezium rule it is usually possible to obtain a more accurate result by first dividing the area under the curve between $x = a$ and $x = b$ into a larger (even) number of strips and then applying (17) to each successive pair. In this way, if $f_0, f_1, f_2, f_3 \ldots f_{n-1}, f_n$ are the values of $f(x)$ at $x = a, a + h, a + 2h, \ldots a + (n + 1)h, a + nh(= b)$, where n is an even integer, then

$$\int_a^b f(x)dx \simeq \frac{h}{3}\{f_0+f_n+4(f_1,+f_3+\ldots+f_{n-1},)+2(f_2,+f_4+\ldots+f_{n-2},)\}. \qquad (20)$$

For example, Simpson's rule with five ordinates (i.e. four strips) is

$$\int_a^b f(x)\,dx \simeq \frac{h}{3}\{f_0 + f_4 + 4(f_1, + f_3) + 2f_2\} \tag{21}$$

(see Fig. 14.2).

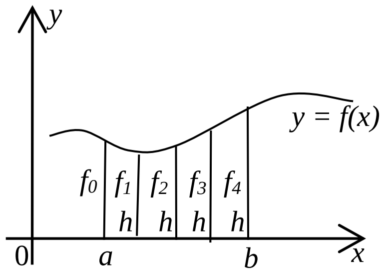

Fig. 3.2

3.3 Application of Simpson's Rule

Example 2: We now consider the numerical evaluation of the integral

$$I = \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - \frac{1}{2}\sin^2\theta)}} \tag{22}$$

using Simpson's rule with five ordinates. This integral is the complete elliptic integral of the first kind $K(1/\sqrt{2})$ whose tabulated value is 1.854. To apply Simpson's rule we now divide the range of integration (0, $\pi/2$) into four parts such that $h = \pi/8$ and evaluate the integrand $f(\theta) = (1 - \frac{1}{2}\sin^2\theta)^{-\frac{1}{2}}$ at the five points θ = 0, $\pi/8$, $\pi/4$, $3\pi/8$ and $\pi/2$.

The values are given below:

θ	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
$f(\theta)$	1.000	1.0387	1.1547	1.3206	1.4142

Hence using (21) we have

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - \frac{1}{2}\sin^2\theta)}}$$

$$\simeq \frac{1}{3}\cdot\frac{\pi}{8}\{1+1.4142+4(1.0387+1.3206) +2\,(1.1547)\} \tag{23}$$

$$= \frac{\pi}{24} \times 14.1608,$$

$$= 1.854. \qquad (24)$$

Example 3: Given the following nine pairs of (x, y) values

x	1	2	3	4	5	6	7	8	9
y	2.061	2.312	2.819	3.106	3.670	4.721	6.103	7.950	9.942

we may easily estimate using (20). Since h =1 we have

$$\int_1^9 y\,dx \simeq \frac{1}{3}\{2.061 + 9.942 + 4(2.312 + 3.106 + 4.721 + 7.950) + 2(2.819 + 3.670 + 6.103)\} = 36.514 \qquad (25)$$

3.4 Series Expansion Method

When a function *f(x)* can be expanded as a power series in x, term-by-term integration is permissible (see Chapter 5, 5,10) and the evaluation of $\int f(x)dx$ is reduced to the summation of a series. This is illustrated by the following examples.

Example 4: To evaluate

$$I = \int_0^1 \frac{\sin x}{x} dx \qquad (26)$$

we use the Maclaurin expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \qquad (27)$$

and write

$$I = \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right) dx \qquad (28)$$

$$= \left[x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{35280} + \ldots\right]_0^1 \qquad (29)$$

$$= \left[1 - \frac{1}{18} + \frac{1}{600} \ldots\right] \simeq 0.946. \qquad (30)$$

Greater accuracy maybe obtained by summing more terms of the series.

Example 5: To evaluate

$$I = \int_0^2 \sqrt{(8 + x^3)\,dx} \tag{31}$$

we expand the integrand by the binomial theorem to give

$$I = 2\sqrt{2}\int_0^2 \sqrt{\left\{1 + \frac{x}{2}^{\,3}\right\}}dx \tag{32}$$

$$= 2\sqrt{2}\int_0^2 \left\{1 + \frac{1}{2}\,\frac{x}{2}^{\,3} + \frac{1}{2}\;-\frac{1}{2}\;\frac{1}{2!}\;\frac{x}{2}^{\,5} + \ldots\right\}dx. \tag{33}$$

Hence integrating term-by-term we have

$$I = 2\sqrt{2}\left[x + \frac{x^4}{64} - \frac{x^7}{3584} + \ldots\right]_0^2 \tag{34}$$

$$= 2\sqrt{2}\left[2 + \frac{1}{4} - \frac{1}{28} + \ldots\right] \simeq 6.25. \tag{35}$$

Example 6: The elliptic integral discussed in Example 2 may be also be evaluated by the series expansion method. To do this we expand the integrand by the binomial theorem to give

$$I = \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - \frac{1}{2}\sin^2\theta)}} \tag{36}$$

$$= \int_0^{\pi/2} \left\{1 + \;-\frac{1}{2}\left(-\frac{\sin^2\theta}{2}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{\sin^2\theta}{2}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{\sin^2\theta}{2}\right)^3 + \ldots\right\}d\theta. \tag{37}$$

Hence, integrating term-by-term, and using Wallis's formula

$$\int_0^2 \sin^n dx = \frac{(n-1)(n-3)(n-5)\ldots 3.1}{n(n-2)(n-4)\ldots 4.2}\cdot\frac{\pi}{2}, \tag{38}$$

where *n* is an even integer, we have

$$I = \frac{\pi}{2}\; 1 + \frac{1}{8} + \frac{9}{256} + \frac{25}{2048} + \ldots\;. \tag{39}$$

Taking the first four terms only, we find $I \simeq 1.843$, which is in close agreement with the exact value of 1.845 (to three places of decimals).

In this example, the term-by-term integration should really be justified since the series in (37) is not a power series in θ. However, as this requires the concept of uniform convergence in Unit 3 we shall accept the validity of it here without proof.

4.0 CONCLUSION

Numerical application to integration has made calculation to be very easy. More of these methods will be considered in the course on Numerical analysis.

5.0 SUMMARY

You have studied the application of trapezoid and Simpson's rule to numerical integration of functions. Of particular importance is solution of elliptic equation, which proved to be very difficult to be integrated analytically.

The method should be learn and demonstrated by anybody who is interested in further research in mathematics.

6.0 TUTOR-MARKED ASSIGNMENT

1) Evaluate the following integrals

a) $\int_0^{\pi/2} \sqrt{\sin\theta)\,d\theta}$

b) $1/\pi \int_0^{\pi/2} \sqrt{4 + \sin^2 \theta d\theta}$

c) $\int_2^3 \frac{dx}{1 + x^4}$

d) $\int_0^{\pi} \sqrt{(3 + \cos\theta)\,d\theta}$

2) Using Simpson's rule estimate

$\int_0^2 y dx$

from the pair of (x, y) values

X	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Y	1.31	2.41	3.04	2.97	2.76	1.80	.075	0.13	0.01

7.0 REFERENCES/FURTHER READING

G. Gstephenson (1977): Mathematical Methods for Science Students

K. A Stroud and Dexter J. Booth(2001): Engineering Mathematics 5th Edition. Palgrave.