NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: FMT 312

COURSE TITLE: MATHEMATICAL PROGRAMMING II

MATHEMATICAL PROGRAMMING II

FMT 312

Course Guide

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CONTENT

INTRODUCTION

You are holding in your hand the course guide for FMT 312 (Linear Programming II). The purpose of the course guide is to relate to you the basic structure of the course material you are expected to study. Like the name 'course guide' implies, it is to guide you on what to expect from the course material and at the end of studying the course material.

COURSE CONTENT

Non – linear programming, quadratic programming Kuhntucker methods, optimality criteria simple variable optimization. Multivariable techniques, Gradient methods.

COURSE AIM

The aim of the course is to bring to your cognizance the different methods of solving (Non-LPP) thus Non-Linear programming models in Finance as mentioned in the course content to handle Financial problems via the use of Statistics and calculations.

COURSE OBJECTIVES

At the end of studying the course material, among other objectives, you should be able to:

(i) Define continuous functions, differentiability and continuous differentiable function in Rn.

(ii) Define and use the concept of partial derivatives and directional derivatives.

(iii) Find Higher order Derivatives of a function defined on a subset S of Rn.

(iv) Define quadratic forms and Definiteness.

(v) Identify definiteness and semidefiniteness.

COURSE MATERIAL

The course material package is composed of:
The Course Guide
The study units
Self-Assessment Exercises
Tutor Marked Assignment
References/Further Reading

THE STUDY UNITS

There are two modules and four units in this course material.

These study units are as listed below:

MODULE I

CLASSICAL OPTIMIZATION THEORY IN RN

UNIT I

Basic Concepts of Rn

UNIT 2

Optimization in Rn

UNIT 3

MODULE II

Unconstrained Optimization

UNIT 4

Constrained Optimization

TUTOR MARKED ASSIGNMENTS

The Tutor Marked Assignments (TMAs) at the end of each unit are designed to test your knowledge and application of the concepts learned. Besides the preparatory TMAs in the course material to test what has been learnt, it is important that you know that at the end of the course, you must have done your examinable TMAs as they fall due, which are marked electronically. They make up to 30 percent of the total score for the course.

SUMMARY

Having gone through this course, you now know

(i) A Typical Optimization Problem is

Minimize(or Maximize) f(x) Subject to: $x \in D$

where $f : D \subset R \rightarrow R$ is called the objective function and D is called the constraint set.

(ii) Optimization problems are of two types, namely Constrained and Unconstrained Prob- lems. It is constrained if the constraint set D is made up of a set of inequalities and/or equations

(iii) If for example the problem min(or max) f(x) subject to $x \in D$

that f is continuous and D is a bounded and closed subset ofRn, the there exist a solution for the problem. This is the Weierstrass Existence theorem theorem. (iv) A real valued function $f : Rn \rightarrow R$ is coercive if you have

$\lim_{kxk \rightarrow +\infty}$

$f(x) = +\infty$.

(v) If f is continuous and coercive on a closed set $D \subset R$ then there exist $\bar{x} \in D$ such that $f(\bar{x}) \leq f(x)$ for all $x \in D$.

(ii) the existence theorems for solution of an optimization problem.

Good luck.

FMT 313
MATHEMATICAL PROGRAMMING II

Prof.U.A. OSISIOGU

CONTENTS

Module I

CLASSICAL OPTIMIZATION THEORY IN R^N

UNIT 1

BASIC CONCEPTS OF R^N

1.1 Introduction

In this unit and subsequent units, you shall be considering another aspect of optimization problems, different from the linear programming problem you have seen in previous units. The theorems you shall develop here are more general to any given mathematical programming in which the objective function $f : S \subset R^n \to R$ defined on a subset S of R^n is nonlinear. Also the constraints may or may not be linear in the decision variables and the non-negativity condition is also relaxed.

For a better understanding of optimization in R^n, you shall, in this unit, be introduced to some basic concepts and notions of the space R^n (also known as the *real n-space*). These notions, can also be referred to as the topology of R^n. Thus, you shall be considering notions like, Continuous functions, differentiability, partial derivatives, directional derivatives and higher order derivatives. You will also consider quadratic forms: definite and semidefinite matrices and also see some results.

1.2 Objectives

At the end of this unit, you should be able to

(i) Define continuous functions, differentiability and continuous differentiable function in R^n.

(ii) Define and use the concept of partial derivatives and directional derivatives.

(iii) Find Higher order Derivatives of a function defined on a subset S of R^n.

(iv) Define quadratic forms and Definiteness.

(v) Identify definiteness and semidefiniteness.

1.3 Functions

Let S, T be subsets of R^n and R^l, respectively. A *function* $\boldsymbol{f}$ from S to T denoted by $\boldsymbol{f} : S \to T$, is a rule that associates with each element of S, one and only one element of T. The set S is called the *domain* of the function $\boldsymbol{f}$, and the set T is the *range* of the function $\boldsymbol{f}$.

1.3.1 Continuous Functions

Definition 1.3.1 *Let $\boldsymbol{f} : S \to T$, where $S \subset R^n$ and $T \subset R^l$. Then, $\boldsymbol{f}$ is said to be* ***continuous*** *at $x \in S$ if for all > 0, there exists a $\delta > 0$ such that $y \in S$ and $d(x, y) < \delta$ implies that $d(\boldsymbol{f}(x), \boldsymbol{f}(y)) <$. (Note that $d(x, y)$ is the distance between x and y in R^n, while $d(\boldsymbol{f}(x), \boldsymbol{f}(y))$ is the distance in R^l.)*

Another way you can define continuous function is by using sequences.

Definition 1.3.2 *The function $\boldsymbol{f} : S \to T$ is continuous at $x \in S$ if for all sequences $\{x_k\}$ such that $x_k \in S$ for all k, and $\lim_{k\to\infty} x_k = x$, then $\lim_{k\to\infty} \boldsymbol{f}(x_k) = \boldsymbol{f}(x)$.*

Intuitively, $\boldsymbol{f}$ is continuous at x if the value of $\boldsymbol{f}$ at any point y that is "close" to x is a good approximation of the value of $\boldsymbol{f}$ at x.

Definition 1.3.3 ***(Discontinuous Function)*** *$\boldsymbol{f} : S \to T$ is called* ***discontinous*** *at $x \in S$ if it is not continuous at x.*

Example 1.3.1 ***(Continuous function)*** The identity function $\boldsymbol{f}(x) = x$ for all $x \in R$ is continuous at each $x \in R$

Example 1.3.2 The function $\boldsymbol{f} : R \to R$ given by

$$\boldsymbol{f}(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

is continuous everywhere except at $x = 0$. At $x = 0$, every open ball $B(x, \delta)$ with center x and radius $\delta > 0$ contains at least one point $y > 0$. At all such points, $\boldsymbol{f}(y) = 1 > 0 = \boldsymbol{f}(x)$, and this approximation does not get better, no matter how close y gets to x (i.e., no matter how small you take δ to be).

Definition 1.3.4 *A function $\boldsymbol{f} : S \to T$ is said to be continuous on S if it is continuous at each point in S.*

Observe that if $\boldsymbol{f} :\subset R^n \to R^l$, then $\boldsymbol{f}$ consists of l "component functions" $(\boldsymbol{f}^1, \ldots, \boldsymbol{f}^l)$, i.e., there are functions $\boldsymbol{f}^i : S \to R$, $i = 1, \ldots, l$, such that for each $x \in S$, you have $\boldsymbol{f}(x) = (\boldsymbol{f}^1(x), \ldots, \boldsymbol{f}^l(x))$.

Proposition 1.3.1 *$\boldsymbol{f}$ is continuous at $x \in S$ (resp. $\boldsymbol{f}$ is continuous on S) if and only if each $\boldsymbol{f}^i$ is continuous at x (resp. if and only if each $\boldsymbol{f}^i$ is continuous on S).*

Theorem 1.3.1 *A function $\boldsymbol{f} : S \subset R^n \to R^l$ is continuous at a point $x \in S$ if and only if for all open set $V \subset R^l$ such that $\boldsymbol{f}(x) \in V$, there is an open set $U \subset R^n$ such that $x \in U$, and $\boldsymbol{f}(z) \in V$ for all $z \in U \cap S$.*

Proof. Suppose $\boldsymbol{f}$ is continuous at x, and V is an open set in R^l containing $\boldsymbol{f}(x)$. Suppose, by contradiction, that the theorem was false, so for any open set U containing x, there is $y \in U \cap S$ such that $\boldsymbol{f}(y) \in V$. Let $k \in \{1, 2, 3, \ldots\}$, let U_k be the open ball with center x and radius $1/k$. Let $y_k \in U_k \cap S$ be such that $\boldsymbol{f}(y_k) \in V$. The sequence $\{y_k\}$ is clearly well defined, and since $y_k \in U_k$ for all k, you have $d(x, y_k) < 1/k$ for each k, so $y_k \to x$ as $k \to \infty$. Since $\boldsymbol{f}$ is continuous at x by hypothesis, you also have $\boldsymbol{f}(y_k) \to \boldsymbol{f}(x)$ as $k \to \infty$. However $\boldsymbol{f}(y_k) \in V$ for any k, and since V is open, V^c is closed, so $\boldsymbol{f}(x) = \lim_{k\to\infty} \boldsymbol{f}(y_k) \in V^c$ which contradicts $\boldsymbol{f}(x) \in V$.

Conversely, suppose that for each open set V containing $\boldsymbol{f}(x)$, there is an open set U containing x such that $\boldsymbol{f}(y) \in V$ for all $y \in U \cap S$. You will show that $\boldsymbol{f}$ is continuous at x. Let > 0 be given. Define V to be the open ball in R^l with center $\boldsymbol{f}(x)$ and radius . Then, there exists an open set U containing x such that $\boldsymbol{f}(y) \in V$ for all $y \in U \cap S$. Pick any $\delta > 0$ so that $B(x, \delta) \in U$. Then, by construction, it is true that $y \in S$ and $d(x, y) < \delta$ implies $\boldsymbol{f}(y) \in V$, i.e., that $d(\boldsymbol{f}(x), \boldsymbol{f}(y)) <$. Since > 0 is arbitrary, you have shown precisely that $\boldsymbol{f}$ is continuous at x. ■

As an immediate corollary, you have the following statement, which is usually abbreviated as: "a function is continuous if and only if the inverse image of every open set is open."

Corollary 1.3.1 *A function $\boldsymbol{f} : S \subset R^n \to R^l$ is continuous on S if and only if for each open set $V \subset R^l$, there is an open set $U \subset R^n$ such that $\boldsymbol{f}^{-1}(V) = U \cap S$ where $\boldsymbol{f}^{-1}(V)$ is defined by*

$$\boldsymbol{f}^{-1}(V) = \{x \in S | \boldsymbol{f}(x) \in V\}$$

In particular, if S is an open set in R^n, $\boldsymbol{f}$ is continuous on S if and only if $\boldsymbol{f}^{-1}(V)$ is an open set in R^n for each open set V in R^l.

Finally, some observation. Note that continuity of a function $\boldsymbol{f}$ at a point x is a *local* property, i.e., it relates to the behaviour of $\boldsymbol{f}$ near x. but tells you nothing about the behaviou of $\boldsymbol{f}$ elsewhere. In particular, the continuity of $\boldsymbol{f}$ at x has no implicatioin even for the continuity of $\boldsymbol{f}$ at points "close" to x. Indeed, it is easy to construct functions that are continuous at a given point x, but that are discontinous at *every* neighbourhood of x. It is also important to note that, in general, functions need not be continuous at even a single point in their domain. Consider $\boldsymbol{f} : R_+ \to R_+$ given by $\boldsymbol{f}(x) = 1$, if x is a rational number, and $\boldsymbol{f}(x) = 0$, otherwise. This function is discontinuous everywhere on R_+.

1.3.2 Differentiable and Continuously Differentiable Functions

Throughout this subsection, S will denote an open set in R^n

Definition 1.3.5 ***(Differentiability)*** *A function* $\boldsymbol{f} : S \to R^m$ *is said to be differentiable at a point* $x \in S$ *if there exists an* $m \times n$ *matrix* A *such that for all* > 0, *there is* $\delta > 0$ *such that* $y \in S$ *and* $x - y < \delta$ *implies*

$$\boldsymbol{f}(x) - \boldsymbol{f}(y) - A(x - y) < x - y$$

.

Equivalently, $\boldsymbol{f}$ *is differentiable at* $x \in S$ *if*

$$\lim_{y \to x} \left(\frac{\boldsymbol{f}(y) - \boldsymbol{f}(x) - A(y - x)}{y - x} \right) = 0$$

(The notation "$y \to x$" is shorthand for " for all sequences $\{y_k\}$ such that $y_k \to x$.")

The matrix A in this case is called *derivative of* $\boldsymbol{f}$ *at* x and is denoted $D\boldsymbol{f}(x)$. Figure 1.1 provides a graphical illustration of the derivative. In keeping with standard practice, you shall, in the sequel, denote $D\boldsymbol{f}(x)$ by $\boldsymbol{f}'(x)$ whenever $n = m = 1$, i.e., whenever $S \subset R$ and $\boldsymbol{f} : S \to R$.

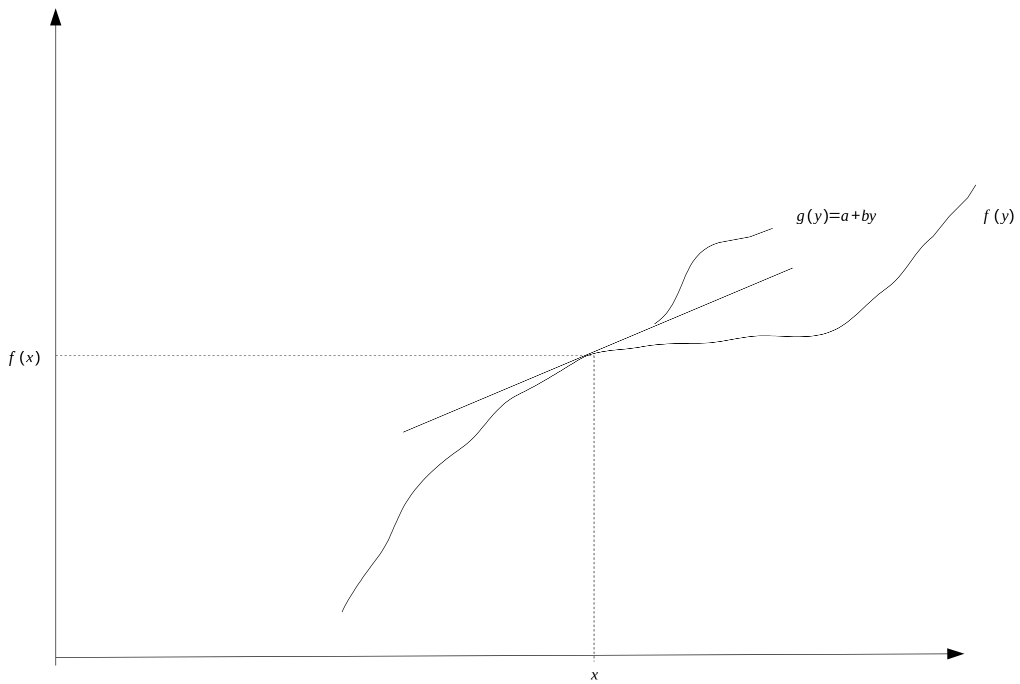

Figure 1.1: The Derivative

Remark 1.3.1 *The definition of the derivative* Df *may be motivated as follows. An* ***affine function*** *from* R^n *to* R^m *is a function* g *is of the form*

$$g(y) = Ay + b,$$

where A is an $m \times n$ matrix, and $b \in \mathbb{R}^m$. (When $b = 0$, the function g is called linear.*) Intuitively, the* derivative *of f at a point $x \in S$ is the* best affine approximation *to f at x, i.e., the best approximation of f around the point x by an affine function g. Here, "best" means that the ratio*

$$\left(\frac{f(y) - g(y)}{y - x} \right)$$

goes to zero as $y \to x$. Since the values of f and g must coincide at x (otherwise g would be hardly be a good approximation to f at x), you must have $g(x) = Ax + b = f(x)$, or $b = f(x) - Ax$. Thus, you may write this approximating function g as

$$g(y) = Ay - Ax + f(x) = A(y - x) + f(x).$$

Given this value for $g(y)$, the task of identifying the best affine approximation to f at x now amounts to identifying a matrix A such that

$$\left(\frac{f(y) - g(y)}{y - x} \right) = \left(\frac{f(y) - (A(y - x)) + f(x)}{y - x} \right) \to 0 \text{ as } y \to x.$$

This is precisely the definition of the derivative you have given.

If f is differentiable at all points in S, then f is said to be *differentiable on S*. When f is differentiable on S, the derivative Df itself forms a function from S to $\mathbb{R}^{m\times n}$. If $Df : S \to \mathbb{R}^{m\times n}$ is a continuous function, then f is said to be *continuously differentiable* on S, and you write f is C^1.

The following observations are immediate from the definitions. A function $f : S \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in S$ if and only if each of the m componet functions $f^i : S \to \mathbb{R}$ of f is differentiable at x, in which case you have $Df(x) = (Df^1(x), \ldots, Df^m(x))$. Moreover, f is C^1 on S if and only if each f^i is C^1 on S.

The difference between differentiability and continuous differentiability is non-trivial. The following example shows that a function may be differentiable everywhere, but may still not be continuously differentiable.

Example 1.3.3 Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x = 0. \end{cases}$$

For $x = 0$, you have

$$f'(x) = 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right).$$

Since $|\sin(\cdot)| \leq 1$ and $|\cos(\cdot)| \leq 1$, but $(2/x) \to \infty$ as $x \to 0$, it is clear that the limit as $x \to 0$ of $f'(x)$ is not well defined. However, $f'(0)$ does exist! Indeed,

$$f'(0) = \lim_{x\to 0} \left(\frac{f(x) - f(0)}{x - 0} \right) = \lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right).$$

Since $|\sin(1/x^2)| \leq 1$, you have $|x\sin(1/x^2)| \leq |x|$, so $x\sin(1/x^2) \to 0$ as $x \to 0$. This means $\boldsymbol{f}'(0) = 0$. Thus, $\boldsymbol{f}$ is not C^1 on $\mathbb{R}_+$.

This example notwithstanding, it is true that the derivative of everywhere differentiable function $\boldsymbol{f}$ must possess a minimal amount of continuity. This you shall see in the intermediate value theorem later in this unit.

You shall close this subsection with a statement of two important properties of the derivative.

First, given two functions $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, define their sum $(\boldsymbol{f} + g)$ to be the function from $\mathbb{R}^n$ to $\mathbb{R}^m$ whose value at any $x \in \mathbb{R}^n$ is $\boldsymbol{f}(x) + g(x)$.

Theorem 1.3.1 *If $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are both differentiable at a point $x \in \mathbb{R}^n$, so is $(\boldsymbol{f} + g)$ and, in fact,*

$$D(\boldsymbol{f} + g)(x) = D\boldsymbol{f}(x) + Dg(x).$$

Proof. Obvious from the definition of differentiability. ■

Next, given functions $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^k \to \mathbb{R}^n$, define, their *composition* $\boldsymbol{f} \circ h$ to be the function from $\mathbb{R}^k$ to $\mathbb{R}^m$ whose value at any $x \in \mathbb{R}^k$ is given by $\boldsymbol{f}(h(x))$, that is, by the value of $\boldsymbol{f}$ evaluated at $h(x)$.

Theorem 1.3.2 *Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^k \to \mathbb{R}^n$. Let $x \in \mathbb{R}^k$. If h is differentiable at x, and $\boldsymbol{f}$ is differintiable at $h(x)$, the $\boldsymbol{f} \circ h$ is itself differetiable at x, and its derivative may be obtained throughout the "chain rule" as:*

$$D(\boldsymbol{f} \circ h)(x) = D\boldsymbol{f}(h(x))Dh(x).$$

Proof. See Rudin (1976, theorem 9.15, p.214). ■

Theorems 1.3.1 and 1.3.2 are only one-way implications. For instance, while the differentiability of $\boldsymbol{f}$ and g at x implies the differentiability of $(\boldsymbol{f} + g)$ at x, $(\boldsymbol{f} + g)$ can be differentiable everywhere (even C^1) without $\boldsymbol{f}$ and g being differentiable anywhere. For an example, let $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}$ be given by $\boldsymbol{f}(x) = 1$ if x is rational, and $\boldsymbol{f}(x) = 0$ otherwise, and let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 0$ if x is rational, and $g(x) = 1$ otherwise. Then, $\boldsymbol{f}$ and g are discontinuous everywhere, so are certainly not differentiable anywhere. However, $(\boldsymbol{f} + g)(x) = 1$ for all x, so $(\boldsymbol{f} + g)'(x) = 0$ at all x, meaning $(\boldsymbol{f} + g)$ is C^1. Similarly, the differeintiability of $\boldsymbol{f} \circ h$ has no implications for the differentiability of $\boldsymbol{f}$ at $h(x)$ or the differentiability of h at x.

1.3.3 Partial Derivatives and Differentiability

Definition 1.3.6 *Let $\boldsymbol{f} : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$ is an open set. Let e_j denote the vector in $\mathbb{R}^n$ that has a 1 in the $\boldsymbol{j} -$ th place and zeros elsewhere ($\boldsymbol{j} = 1, \ldots, n$). Then the $\boldsymbol{j} -$ th* ***partial derivative*** *of $\boldsymbol{f}$ is said to exist at a point x if there is a number $\partial\boldsymbol{f}(x)/\partial x_j$ such that*

$$\lim_{t\to 0} \left(\frac{\boldsymbol{f}(x + te_j) - \boldsymbol{f}(x)}{t}\right) = \frac{\partial \boldsymbol{f}}{\partial x_j}(x)$$

Among the more pleasant facts of life are the following:

Theorem 1.3.3 *Let $\boldsymbol{f} : S \to R$, where $S \subset R^n$ is open.*

1. *If $\boldsymbol{f}$ is differentiable at x, then all partials $\partial \boldsymbol{f}(x)/\partial x_j$ exist at x, and* $D\boldsymbol{f}(x) = [\partial \boldsymbol{f}(x)/\partial x_1, \ldots, \partial \boldsymbol{f}(x)/\partial x_n]$
2. *If all the partials $\partial \boldsymbol{f}(x)/\partial x_j$ exist and are continuous at x, then $D\boldsymbol{f}(x)$ exists and* $D\boldsymbol{f}(x) = [\partial \boldsymbol{f}(x)/\partial x_1, \ldots, \partial \boldsymbol{f}(x)/\partial x_n]$
3. *$\boldsymbol{f}$ is C^1 on S if and only if all partial derivatives of $\boldsymbol{f}$ exist and are continuous on S.*

Proof. See Rudin (1976, Theorem 9.21, p219). ■

Thus, to check if $\boldsymbol{f}$ is C^1, you only need figure out if (a) the partial derivatives exist on S, and (b) if they are all continuous on S. On the other hand, the requirement that the partials not only exist but be continuous at x is very important for the coincidence of the vector of partials with $D\boldsymbol{f}(x)$. In the absence of this condition, all partials could exist at some point without the function itself being differentiable at that point. Consider the following example:

Example 1.3.4 Let $\boldsymbol{f} : R^2 \to R$ be given by $\boldsymbol{f}(0,0) = 0$, and for $(x, y) = (0, 0)$

$$\boldsymbol{f}(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}.$$

You will show that $\boldsymbol{f}$ has all partial derivatives everywhere (including at $(0, 0)$), but that these partials are not continuous at $(0, 0)$. Then you have to show that $\boldsymbol{f}$ is differentiable at $(0, 0)$.

☞ ***Solution.*** Since $\boldsymbol{f}(x, 0) = 0$ for any $x = 0$, it is immediate that for all $x = 0$,

$$\frac{\partial \boldsymbol{f}}{\partial y}(x, 0) = \lim_{\hat{y}\to 0} \frac{\boldsymbol{f}(x, \hat{y}) - \boldsymbol{f}(x, 0)}{\hat{y}} = \lim_{\hat{y}\to 0} \frac{x}{\sqrt{x^2 + y^2}} = 1.$$

Similarly, at all points of the form $(0, y)$ for $y = 0$, you have $\partial \boldsymbol{f}(0, y)/\partial x = 1$. However, note that

$$\frac{\partial \boldsymbol{f}}{\partial x}(0, 0) = \lim_{x\to 0} \frac{\boldsymbol{f}(x, 0) - \boldsymbol{f}(0, 0)}{x} = \lim_{x\to 0} \frac{0 - 0}{x} = 0,$$

so $\partial \boldsymbol{f}(0, 0)/\partial x$ exists at $(0, 0)$, but is not the limit of $\partial \boldsymbol{f}(0, y)/\partial x$ as $y \to 0$. Similarly, you also have $\partial \boldsymbol{f}(0, 0)/\partial y = 0 = 1 = \lim_{x\to 0} \partial \boldsymbol{f}(x, 0)/\partial y$.

Suppose $\boldsymbol{f}$ were differentiable at $(0, 0)$. Then, the derivatives $D\boldsymbol{f}(0, 0)$ must conicide with the vector of partials at $(0, 0)$ so you must have $D\boldsymbol{f}(0, 0) = (0, 0)$. However, from the definition of the derivative, you must also have

$$\lim_{(x,y)\to(0,0)} \frac{\boldsymbol{f}(x, y) - \boldsymbol{f}(0, 0) - D\boldsymbol{f}(0, 0) \cdot (x, y)}{\|(x, y) - (0, 0)\|} = 0$$

but this is impossible if $D\boldsymbol{f}(0, 0) = 0$. To see this, take any point (x, y) of the form (a, a) for some $a > 0$, and note that every neighbourhood of $(0, 0)$ contains at least one such point. Since $\boldsymbol{f}(0, 0) = 0$, $D\boldsymbol{f}(0, 0) = (0, 0)$, and $\|(x, y)\| = \sqrt{x^2 + y^2}$, it follows that

$$\frac{\boldsymbol{f}(a, a) - \boldsymbol{f}(0, 0) - D\boldsymbol{f}(0, 0) \cdot (a, a)}{\|(a, a) - (0, 0)\|} = \frac{a^2}{2a^2} = \frac{1}{2}$$

so the limit of this fraction as $a \to 0$ cannot be zero. ✍

Intuitively, the feature that drives this example is that in looking at the partial derivative of f with respect to (say) x at a point (x, y), you are moving along only the line through (x, y) parallel to the x-axis (see the line denoted l_1 in Figure 1.2). Similarly, the partial with derivative with respect to y involves holding the x variable fixed, and moving only on the line through (x, y) parallel to the y-axis (see the line denoted l_2 in Figure 1.2). On the other hand, in looking at the derivative Df, both the x and y variables are allowed to vary *simultaneously* (for instance, along the dotted curve in Figure 1.2).

Lastly, it is worth stressing that although a function must be continuous in order to be differentiable (this is easy to see from the definitions), there is no implication in the other direction whatsoever. Extreme examples exist of functions which are continuous on all of R, but fail to be differentiable at even a single point. Such functions are by no means pathological; they play, for instance, a central role in the study of Brownian motion in probability theory (with probability one, a Brownian motion path is everywhere continuous and nowhere differentiable).

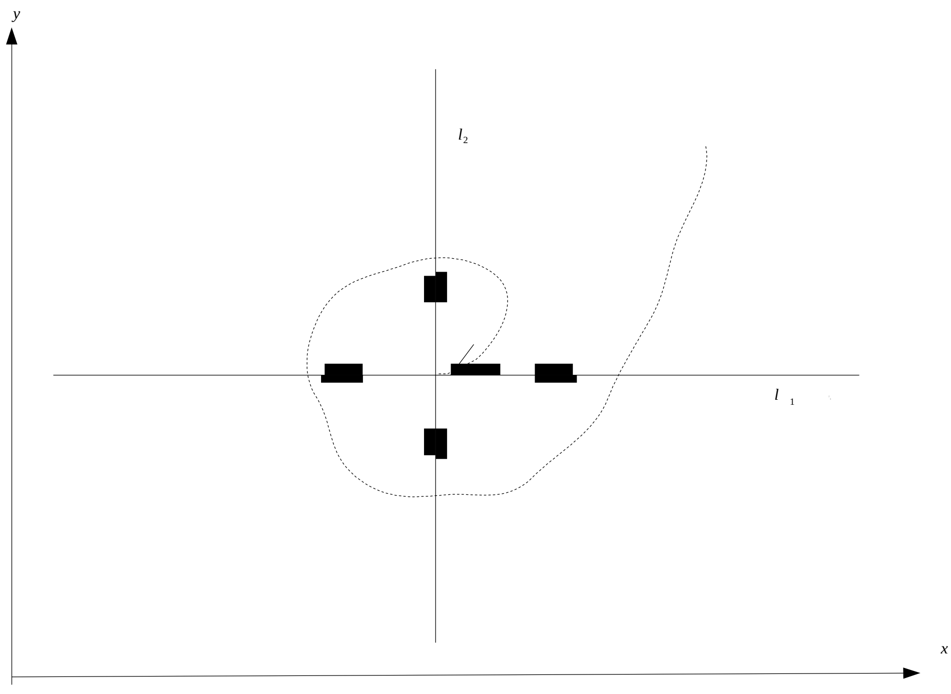

Figure 1.2: Partial Derivatives and Differentiability

1.3.4 Directional Derivatives and Differentiability

Let $f : S \to R$, where $S \subset R^n$ is open. Let x be any point in S, and let $h \in R^n$. The *directional derivative of f at x in the direction h* is defined as

$$\lim_{t \to 0+} \left(\frac{f(x + th) - f(x)}{t} \right)$$

when this limit exists, and is denoted $D\boldsymbol{f}(x; h)$. (The notation $t \to 0+$ is shorthand for $t > 0$, $t \to 0$.)

When the condition $t \to 0+$ is replaced with $t \to 0$, you obtain what is sometimes called the "two-sided directional derivative." Observe that partial derivatives are a special case of two-sided directional derivatives: when $h = e_i$ for some i, the two-sided directional derivative at x is precisely the partial derivative $\partial \boldsymbol{f}(x)/\partial x_i$.

In the privious subsection, it was pointed out that the existence of all partial derivatives at a point x is not sufficient to ensure that $\boldsymbol{f}$ is differentiable at x. It is actually true that no even the existence of *all* two-sided directional derivatives at x implies that $\boldsymbol{f}$ is differentiable at x. However, the following relationship in the reverse direction is easy to show.

Theorem 1.3.4 *Suppose $\boldsymbol{f}$ is differentiable at $x \in S$. Then, for any $h \in R^n$, the (one-sided) directional derivative $D\boldsymbol{f}(x; h)$ of $\boldsymbol{f}$ at x in the direction h exists, and, in fact, you have $D\boldsymbol{f}(x; h) = D\boldsymbol{f}(x) \cdot h$.*

An immediate corollary is

Corollary 1.3.2 *If $D\boldsymbol{f}(x)$ exists, then $D\boldsymbol{f}(x; h) = - D\boldsymbol{f}(x; -h)$.*

Remark 1.3.2 *What is the relationship between $D\boldsymbol{f}(x)$ and the two-sided directional derivative of $\boldsymbol{f}$ at x in an arbitrary direction h?*

1.3.5 Higher Order Derivatives

Let $\boldsymbol{f}$ be a function from $S \subset R^n$ to R, where S is an open set. Throughout this subsection, you will assume that $\boldsymbol{f}$ is differentiable on all of S, so that the derivative $D\boldsymbol{f} = [\partial \boldsymbol{f}/\partial x_1, \ldots, \partial \boldsymbol{f}/\partial x_n]$ itself defines a function from S to R^n.

Suppose now that there is $x \in S$ such that the derivative $D\boldsymbol{f}$ is itself differentiable at x, i.e., such that for each i, the function $\partial \boldsymbol{f}/\partial x_i : S \to R$ is differentiable at x. Denote the partial of $\partial \boldsymbol{f}/\partial x_i$ in the direction e_j at x by $\partial^2 \boldsymbol{f}(x)/\partial x_j \partial x_i$, if $i = j$, and $\partial^2 \boldsymbol{f}(x)/\partial x_i^2$, if $i = j$. Then, you say that $\boldsymbol{f}$ is *twice-differentiable* at x, with second derivative $D^2 \boldsymbol{f}(x)$, where

$$D^2 \boldsymbol{f}(x) = \begin{pmatrix} \dfrac{\partial^2 \boldsymbol{f}(x)}{\partial x_1^2} & \cdots & \dfrac{\partial^2 \boldsymbol{f}(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 \boldsymbol{f}(x)}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 \boldsymbol{f}(x)}{\partial x_n^2} \end{pmatrix}$$

Once again, you shall follow standard practice and denote $D^2 \boldsymbol{f}(x)$ by $\boldsymbol{f}''(x)$ whenever $n = 1$ (i.e., if $S \subset R$).

If $\boldsymbol{f}$ is twice-differentiable at each x in S, you say that $\boldsymbol{f}$ is twice-differentiable on S. When $\boldsymbol{f}$ is twice-differentiable on S, and for each $i, j = 1, \ldots, n$ the cross-partial $\partial^2 \boldsymbol{f}/\partial x_i \partial x_j$ is a *continuous* function from S to R, you say that $\boldsymbol{f}$ is *twice continuously differentiable* on S, and you write $\boldsymbol{f}$ is C^2.

When $\boldsymbol{f}$ is C^2, the second-derivative $D^2\boldsymbol{f}$, which is also called the matrix of cross-partials (or the *hessian* of $\boldsymbol{f}$ at x), has the following useful property:

Theorem 1.3.5 *If* $\boldsymbol{f} : D \to \mathrm{R}^n$ *is a* C^2 *function,* $D^2\boldsymbol{f}$ *is a symmetric matrix, i.e., you have*

$$\frac{\partial^2 \boldsymbol{f}}{\partial x_i \partial x_j}(x) = \frac{\partial^2 \boldsymbol{f}}{\partial x_i \partial x_j}(x)$$

for all $i, j = 1, \ldots, n$ *and for all* $x \in D$.

Proof. See Rudin (1976, Corollary to Theorem 9.41, p.236). ■

For an example where the symmetry of $D^2\boldsymbol{f}$ fails because it fails to be continuous, see the Tutor Marked Assignemts(TMAs).

The condition that the partials should be continuous for $D^2\boldsymbol{f}$ to be a symmetric matrix can be weakened a little. In particular, for

$$\frac{\partial^2 \boldsymbol{f}}{\partial x_j \partial x_k}(y) = \frac{\partial^2 \boldsymbol{f}}{\partial x_k \partial x_j}(y)$$

to hold, it suffices just that (a) the partials $\partial \boldsymbol{f} / \partial x_j$ and $\partial \boldsymbol{f} / \partial x_k$ exist everywhere on D and (b) that one of the cross-partials $\partial^2 \boldsymbol{f} / \partial x_j \partial x_k$ or $\partial^2 \boldsymbol{f} / \partial x_k \partial x_j$ exist everywhere on D and be continuous at y.

Still higher derivatives (third, fourth, etc.) may be defined for a function $\boldsymbol{f} : \mathrm{R}^n \to \mathrm{R}$. The underlying idea is simple: for instance, a function is thrice-differentiable at a point x if all the component functions of its second-derivative $D^2\boldsymbol{f}$ (i.e., if all the cross-partial functions $\partial^2 \boldsymbol{f} / \partial x_i \partial x_j$) are themselves differentiable at x; it is C^3 if all these component functions are continuously differentiable, etc. On the other hand, the notation becomes quite complex unless $n = 1$ (i.e., $\boldsymbol{f} : \mathrm{R} \to \mathrm{R}$), and you do not have any use in this book for derivatives beyond the second, so you will not attempt formal definitions here.

1.4 Quadratic Forms: Definite and Semidefinite Matrices

1.4.1 Quadratic Forms and Definiteness

Definition 1.4.1 *A* ***quadratic form*** *on* R^n *is a function* g_A *on* R^n *of the form*

$$g_A(x) = x^{\mathbf{t}} A x = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

where $A = (a_{ij})$ *is any symmetric* $n \times n$ *matrix.*

Since the quadratic form g_A is completely specified by the matrix A, you henceforth refer to A itself as the quadratic form. your interest in quadratic forms arises from the fact that if $\boldsymbol{f}$ is a C^2

function, and z is a point in the domain of f, then the matrix of second partials $D^2 f\ (z)$ defines a quadratic form (this follows from Theorem 1.3.5 on the symmetry property of $D^2 f$ for a C^2 function f).

Definition 1.4.2 *A quadratic form A is said to be*

1. ***positive definite*** *if you have* $x^{\mathbf{t}}Ax > 0$ *for all* $x \in \mathbb{R}^n$, $x = 0$.
2. ***positive semidefinite*** *if you have* $x^{\mathbf{t}}Ax \geq 0$ *for all* $x \in \mathbb{R}^n$, $x = 0$.
3. ***negative definite*** *if you have* $x^{\mathbf{t}}Ax < 0$ *for all* $x \in \mathbb{R}^n$, $x = 0$.
4. ***negative semidefinite*** *if you have* $x^{\mathbf{t}}Ax \leq 0$ *for all* $x \in \mathbb{R}^n$, $x = 0$

The terms “non-negative definite” and “nonpositive definite” are often used in place of “positive semidefinite” and “negative semidefinite” respectively.

For instance, the quadratic form A defined by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is positive definite, since for any $x = (x_1, x_2) \in \mathbb{R}^2$, you have $x^{\mathbf{t}}Ax = x_1^2 + x_2^2$, and this quantity is positive whenever $x = 0$. On the other hand, consider the quadratic form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

For any $x = (x_1, x_2) \in \mathbb{R}^2$, you have $x^{\mathbf{t}}Ax = x_1^2$, so $x^{\mathbf{t}}Ax$ can be zero even if $x = 0$. (For example, $x^{\mathbf{t}}Ax = 0$ *if* $x = (0, 1)$.) Thus, A is not positive definite. On the other hand, it is certainly true that you always have $x^{\mathbf{t}}Ax \geq 0$, so A is positive semidefinite.

Observe that there exist matrices A which are neither positive semidefinite nor negative semidefinite, and that do not, therefore, fit into any of the four categories you have identified. Such matrices are called *indefinite quadratic forms.* As an example of an indefinite quadratic form A, consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For $x = (1, 1)$, $x^{\mathbf{t}}Ax = 2 > 0$, so A is not negative semidefinite. But for $x = (-1, 1)$, $x^{\mathbf{t}}Ax = -2 < 0$, so A is positive semidefinite either.

Given a quadratic form A and any $t \in \mathbb{R}$, you have $(tx)^{\mathbf{t}} A(tx) = t^2 x^{\mathbf{t}}Ax$, so the quadratic form has the same sign along lines through the origin. Thus, in particular, A is positive definite (resp. negative definite) if and only if it satisfies $x^{\mathbf{t}}Ax > 0$ (resp. $x^{\mathbf{t}}Ax < 0$) for all x in the unit sphere $C = \{u \in \mathbb{R}^n |\ u = 1\}$. You will use this observation to show that if A is a positive definite (or negative definite) $n \times n$ matrix, so is any other quadratic form B which is sufficiently close to A.

Theorem 1.4.1 *Let A be a positive definite $n \times n$ matrix. Then there is $\gamma > 0$ such that if B is any symmetric $n \times n$ matrix with $|b_{jk} - a_{jk}| < \gamma$ for all $j, k \in \{1, \dots, n\}$, then B is also positive definite. A similar statement holds for negative definite matrices A.*

Proof. You will make use of the Weierstrass Theorem, which will be proved later. The Weierstrass Theorem states that if $K \subset \mathbb{R}^n$ is compact, and $f : K \to \mathbb{R}$ is a continuous function, then f has both maximum and minimum on K, i.e., there exist points k' and k^* in K such that $f(k') \geq f(k) \geq f(k^*)$ for all $k \in K$.

Now, the unit sphere C is clearly compact, and the quadratic form A is continuous on this set. Therefore, by the Weierstrass Theorem, there is $z \in C$ such that for any $x \in C$, you have

$$z^t A z \leq x^t A x.$$

If A is positive definite, then $z^t A z$ must be strictly positive, so there must exists > 0 such that $x^t A x \geq \ > 0$ for all $x \in C$.

Define $\gamma = \ /2n^2 > 0$. Let B be any symmetric $n \times n$ matrix, which is such that $|b_{jk} - a_{jk}| < \gamma$ for all $j, k = 1, \dots, n$. Then for any $x \in C$,

$$\begin{aligned}
|x^t(B - A)x| &= \left|\sum_{j,k=1}^{n} (b_{jk} - a_{jk}) x_j x_k\right| \\
&\leq \sum_{j,k=1}^{n} |b_{jk} - a_{jk}||x_j||x_k| \\
&< \gamma \sum_{j,k=1}^{n} |x_j||x_k| \\
&< \gamma n^2 = \ /2.
\end{aligned}$$

Therefore, for any $x \in C$,

$$x^t B x = x^t A x + x^t (B - A)x \geq \ - \ /2 = \ /2$$

so B is positive definite, and the desired result is establised. ■

A particular implication of this result, which you will use in the study of unconstrained optimization problems, is the following:

Corollary 1.4.1 *If f is a C^2 function such that at some point x, $D^2 f(x)$ is a positive definite matrix, then there is a neighbourhood $B(x, r)$ of x such that for all $y \in B(x, r)$, $D^2 f(y)$ is also a positive definite matrix. A similar statement holds if $D^2 f(x)$ is instead, a negative definite matrix.*

Finally, it is important to point out that Theorem 1.4.1 is no longer true if "positive definite" is replaced with "positive semidefinite." Consider, as a counter example, the matrix A defined by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

You have seen above that A is positive semidefinite (but not positive definite). Pick any $\gamma > 0$. Then, for $= \gamma/2$, the matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & - \end{pmatrix}$$

satisfies $|a_{ij} - b_{ij}| < \gamma$ for all i, j. However, B is not positive semidefinite: for $x = (x_1, x_2)$, you have $x^t B x = x_1^2 - x_2^2$, and this quantity can be negative (for instance, if $x_1 = 0$ and $x_2 = 0$). Thus, there is no neighbourhood of A such that all quadratic forms in that neighbourhood are also positive semidefinite.

1.4.2 Identifying Definiteness and Semidefiniteness

From a practical standpoint, it is of interest to ask: what restrictions on the structure of A are imposed by the requirement that A be a positive (or negative) definite quadratic from? The answers to this questions is provided in this section. These results are, in fact, *equivalence* statements; that is, quadratic forms possess the required definiteness or semidefiniteness property *if and only if* they meet the condition outlined.

The first result deals with positive and negative definiteness. Given an $n \times n$ symmetric matrix A, let A_k denote the $k \times k$ submatrix of A that is obtained when only the first k rows and columns are retained, i.e., let

$$A_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$$

You will refer to A_k as the *k-th natural ordered principal minor* of A.

Theorem 1.4.2 *An $n \times n$ symmetric matrix A is*

1. *negative definite if and only if $(-1)^k |A_k| > 0$ for all $k \in \{1, \ldots, n\}$.*
2. *positive definite if and only if $|A_k| > 0$ for all $k \in \{1, \ldots, n\}$.*

Moreover, a positive semidefinite quadratic form A is positive definite if and only if $|A| = 0$, while a negative semidefinite quadratic form is negative definite if and only if $|A| = 0$.

Proof. See Debreu (1952, Theorem 2, p.296). ■

A natural conjecture is that this theorem would continue to hold if the words "negative definite" and "positive definite" were replaced with "negative semidefinite" and "positive semidefinite," respectively, provided the strict inequalities were replaced with weak ones. *This conjecture is false.* Consider the following example.

Example 1.4.1 Let

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

Then, A and B are both symmetric matrices. Moreover, $|A_1| = |A_2| = |B_1| = |B_2| = 0$, so if the conjecture were true, both A and B would pass the test for positive semidefiniteness, as well as the test for negative semidefiniteness. However, for any $x \in R^2$, $x^{\mathbf{t}}Ax = x_2^2$ and $x^{\mathbf{t}}Bx = -x_2^2$. Therefore, A is positive semidefinite but not negative seimidefinite, while B is negative semidefinite, but not positive semidefinite.

Roughly speaking, the feature driving this counterexample is that, in both the matrices A and B, the zero entries in all but the (2, 2)-place of the matrix make the determinants of order 1 and 2 both zero. In particular, no play is given to the sign of the entry in the (2, 2)-place, which is positive in one case, and negative in the other. On the other hand, an examination of the expression $x^{\mathbf{t}}Ax$ and $x^{\mathbf{t}}Bx$ reveals that in both cases, the sign of the quadratic form is determined precisely by the sign of the (2, 2)-entry.

This problem points to the need to expand the set of submatrices that you are considering, if you are to obtain an analog of Theorem 1.4.2 for positive and negative semidefiniteness. Let an $n \times n$ symmetric matrix A be given, and let $\pi = (\pi_1, \ldots, \pi_n)$ be a permutation of the integers $\{1, \ldots, n\}$. Denote by A^π the symmetric $n \times n$ matrix obtained by applying the permutation π to both the rows and columns of A :

$$A^\pi = \begin{pmatrix} a_{\pi_1\pi_1} & \cdots & a_{\pi_1\pi_n} \\ \vdots & \ddots & \vdots \\ a_{\pi_n\pi_1} & \cdots & a_{\pi_n\pi_n} \end{pmatrix}$$

For $k \in \{1, \ldots, n\}$, let A_k^π denote the $k \times k$ symmetric submatrix of A^π obtained by retaining only the first k rows and columns:

$$A_k^\pi = \begin{pmatrix} a_{\pi_1\pi_1} & \cdots & a_{\pi_1\pi_n} \\ \vdots & \ddots & \vdots \\ a_{\pi_k\pi_1} & \cdots & a_{\pi_k\pi_k} \end{pmatrix}$$

Finally, let Π denote the set of all possible permutations of $\{1, \ldots, n\}$

Theorem 1.4.3 *A symmetric $n \times n$ matrix A is*

1. *positive semidefinite if and only if $|A_k^\pi| \geq 0$ for all $k \in \{1, \ldots, n\}$ and for all $\pi \in \Pi$.*
2. *negative semidefinite if and only if $(-1)^k|A_k^\pi| \geq 0$ for all $k \in \{1, \ldots, n\}$ and for all $\pi \in \Pi$.*

Proof. See Debreu (1952, Theorem 7, p298). ∎

One final remark is important. The symmetry assumptions is crucial to the validity of these results. If it fails, a matrix A might pass all the tests for (say) positive semidefiniteness without actually being positive semidefinite. Here are two examples:

Example 1.4.2 Let

$$A = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

Note that $|A_1| = 1$, and $|A_2| = (1)(1) - (-3)(0) = 1$, so A passes the test for positive definiteness. However, A is not a symmetric matrix, and is not, in fact, positive definite: you have $x^{\mathbf{t}}Ax = x_1^2 + x_2^2 - 3x_1x_2$ which is negative for $x = (1, 1)$.

Example 1.4.3 Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

There are only two possible permutations of the set $\{1, 2\}$, namely, $\{1, 2\}$ itself, and $\{2, 1\}$. This gives rise to four different submatrices, whose determinants you have to consider:

$$[a_{11}],\ [a_{22}],\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ \text{and}\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

You can easily check that the determinants of all four of these are non-negative, so A passes the test for positive semidefiniteness. However, A is not positive semidefinite: you have $x^{\mathbf{t}}Ax = x_1x_2$, which could be positive or negative.

1.5 Some Important Results

This section brings together some results of importance for the study of optimization theory. These are, the separation theorems for convex sets in R^n, consequences of assuming continuity and/or differentiability of real-valued functions defined on R^n and two fundamental results known as the Inverse Function Theorem and the Implicit Function Theorem.

1.5.1 Separation Theorems

Let $p = 0$ be a vector in R^n, and let $a \in R$. The set H defined by

$$H = \{x \in R^n | p \cdot x = a\}$$

is called a *hyperplane* in R^n, and will be denoted $H(p, a)$.

A hyperplane in R^2, for example, is simply a straight line: if $p \in R^2$ and $a \in R$, the hyperplane $H(p, a)$ is simply the set of points (x_1, x_2) that satisfy $p_1x_1 + p_2x_2 = a$. Similary, a hyperplane in R^3 is a plane.

A set D in R^n is said to be bounded by a hyperplane $H(p, a)$ if D lies entirely on one side of $H(p, a)$, i.e., if either

$$p \cdot x \leq a, \text{ for all } x \in D$$

or

$$p \cdot x \geq a, \text{ for all } x \in D$$

If D is bounded by $H(p, a)$ and $D \cap H(p, a) = \emptyset$, then $H(p, a)$ is said to be a *supporting hyperplane* for D.

Example 1.5.1 Let $D = \{(x, y) \in R^2_+ | xy \geq 1\}$. Let p be the vector $(1, 1)$, and let $a = 2$. Then the hyperplane

$$H(p, a) = \{(x, y) \in R^2 | x + y = 2\}$$

bounds D : if $xy \geq 1$ and $x, y \geq 0$, then you must have $(x + y) \geq (x + x^{-1}) \geq 2$.

In fact,
$H(p, a)$ is a supporting hyperplane for D since $H(p, a)$ and D have the point $(x, y) = (1, 1)$ in common.

Two sets D and E in R^n are said to be *separated* by the hyperplane $H(p, a)$ in R^n if D and E lie on opposite sides of $H(p, a)$, i.e., if you have

$$p \cdot y \leq a, \text{ for all } y \in D$$

$$p \cdot z \geq a, \text{ for all } y \in D$$

If D and E are separated by $H(p, a)$ and one of the sets (say, E) consists of just a single point x, you will indulge in a slight abuse of terminology and say that $H(p, a)$ separates the set D and the point x.

A final definition is required before you would state the main results of this section. Given a set $X \subset R^n$, the *closure of* X, denoted X°, is defined to be the intersection of all closed sets containing X, i.e., if

$$\Delta(X) = \{Y \subset R^n | X \subset Y\}$$

then

$$X^\circ = \bigcap_{Y \in \Delta(X)} Y.$$

Intuitively, the closure of X is the "smallest" closed set that contains X. Since the arbitrary intersection of closed sets is closed, X° is closed for any set X°. Note that $X^\circ = X$ if and only if X is itself closed.

The following results deal with the separation of convex sets by hyperplanes. They play a significant role in the study of inequality-constrained optimization problems under convexity restriction.

Theorem 1.5.1 *Let D be a nonempty convex set in R^n, and let x^* be a point in R^n that is not in D. Then, there is a hyperplane $H(p, a)$ in R^n with $p = 0$ which separates D and x^*. You may, if you desire choose p to also satisfy $p = 1$.*

Proof. See Sundaram (1999, Theorem 1.67, p56) ■

Theorem 1.5.2 *Let D and E be convex sets in $\mathbb{R}^n$ such that $D \cap E = \emptyset$. Then, there exists a hyperplane $H(p, a)$ in $\mathbb{R}^n$ which separates D and E. You may, if you desire, choose p to also satisfy $\|p\| = 1$.*

Proof. Let $F = D + (-E)$, where, in obvious notation, $-E$ is the set

$$\{y \in \mathbb{R}^n \mid -y \in E\}.$$

Since D and E are convex sets, F is also convex. You can claim that $0 \notin F$. For if you had $0 \in F$, then there would exist points $x \in D$ and $y \in E$ such that $x - y = 0$. But this implies $x = y$, so $x \in D \cap E$, which contradicts the assumption that $D \cap E$ is empty. Therefore, $0 \notin F$.

By 1.5.1, there exists $p \in \mathbb{R}^n$ such that

$$p \cdot 0 \leq p \cdot z, \quad z \in F.$$

This is the same thing as

$$p \cdot y \leq p \cdot x, \quad x \in D, \ y \in E$$

It follows that $\sup_{y \in E} p \cdot y \leq \inf_{x \in D} p \cdot x$. If $a \in \{\sup_{y \in E} p \cdot y, \inf_{x \in D} p \cdot x\}$, the hyperplane $H(p, a)$ separates D and E.

That p can also be chosen to satisfy $\|p\| = 1$ is established in the same way as in 1.5.1 ■

1.5.2 The Intermediate and Mean Value Theorems

The Intermediate Value Theorem asserts that a continuous real function on an interval assumes all intermediate values on the interval. Figure 1.3 illustrates the result.

Theorem 1.5.3 *(Intermediate Value Theorem)* *Let $D = [a, b]$ be an interval in $\mathbb{R}$ and let $f : D \to \mathbb{R}$ be continuous function. If $f(a) < f(b)$, and if c is a real number such that $f(a) < c < f(b)$, then there exists $x \in (a, b)$ such that $f(x) = c$. A similar statement holds if $f(a) > f(b)$.*

Proof. See Rudin (1976, Theorem 4.23, p.93). ■

Remark 1.5.1 *It might appear at first glance that the intermediate value property actually characterizes continuous functions is and only if for any two points $x_1 < x_2$ and for any real number c lying between $f(x_1)$ and $f(x_2)$, there is $x \in (x_1, x_2)$ such that $f(x) = c$. The Intermediate Value Theorem shows that the "only if" part is true. You can show that the converse, namely the "if" part, is actually false.*

You have seen in Example 1.3.3 that a function may be differentiable everywhere, but may fail to be continuously differentiable. The following result (which may be regarded as an Intermediate Value Theorem for the derivative) states, however, that the derivative must still have some minimal continuity properties, viz., that the derivative must assume all intermediate values. In particular, it shows that the derivative f' of an everywhere differentiable function f

cannot have jump discontinuities.

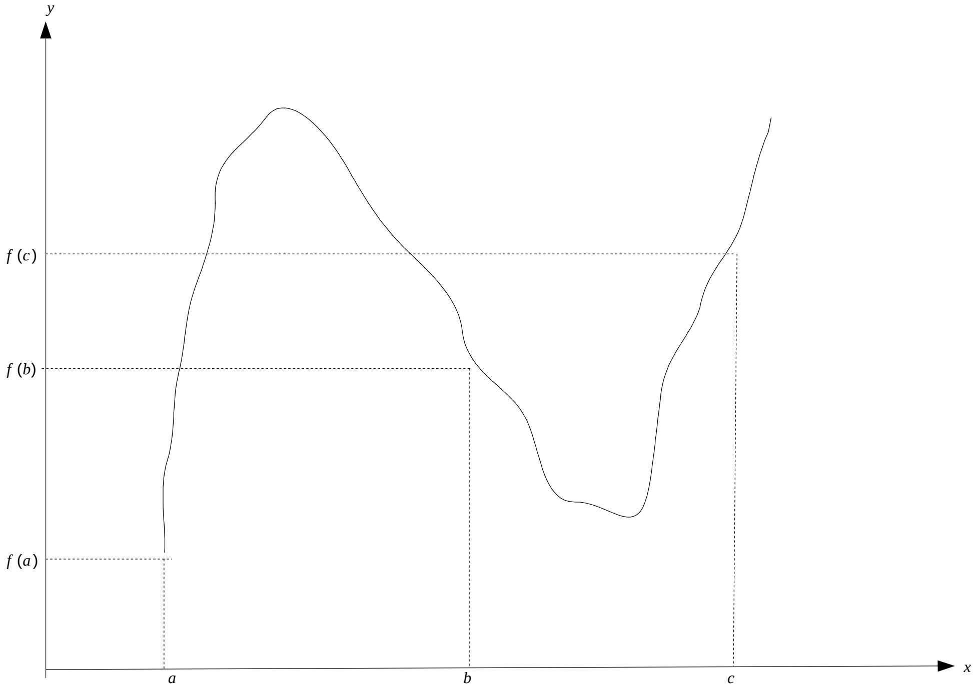

Figure 1.3: The Intermediate Value Theorem

Theorem 1.5.4 ***(Intermediate Value Theorem for the Derivative)*** *Let $D = [a, b]$ be an interval in* R*, and let $f : D \to$ R be a function that is differentiable everywhere on D. If $f'(a) < f'(b)$, and if c is a real number such that $f'(a) < c < f'(b)$, then there is a point $x \in (a, b)$ such that $f'(x) = c$. A similar statement holds if $f'(a) > f'(b)$.*

Proof. See Rudin (1976, Theorem 5.12, p.108) ■

It is very important to emphasize that Theorem 1.5.4 does not assume that f is a C^1 function. Indeed, if f were C^1, the result would be a trivial consequence of the Intermediate Value Theorem, since the derivative f' would then be a continuous function on D.

The next result, the Mean Value Theorem, provides another property that the derivative must satisfy. A graphical representation of this result is provided in Figure 1.4. As with theorem 1.5.4, it is assumed only that f is everywhere differentiable on its domain D, and not that it is C^1.

Theorem 1.5.5 ***(Mean Value Theorem)*** *Let $D = [a, b]$ be an interval in* R*, and let $f : D \to$ R be a continuous function. Suppose f is differentiable on (a, b). Then there exists $x \in (a, b)$ such that*

$$f(b) - f(a) = (b - a)f'(x).$$

Proof. See Rudin (1976, Theorem 5.10, p.108) ■

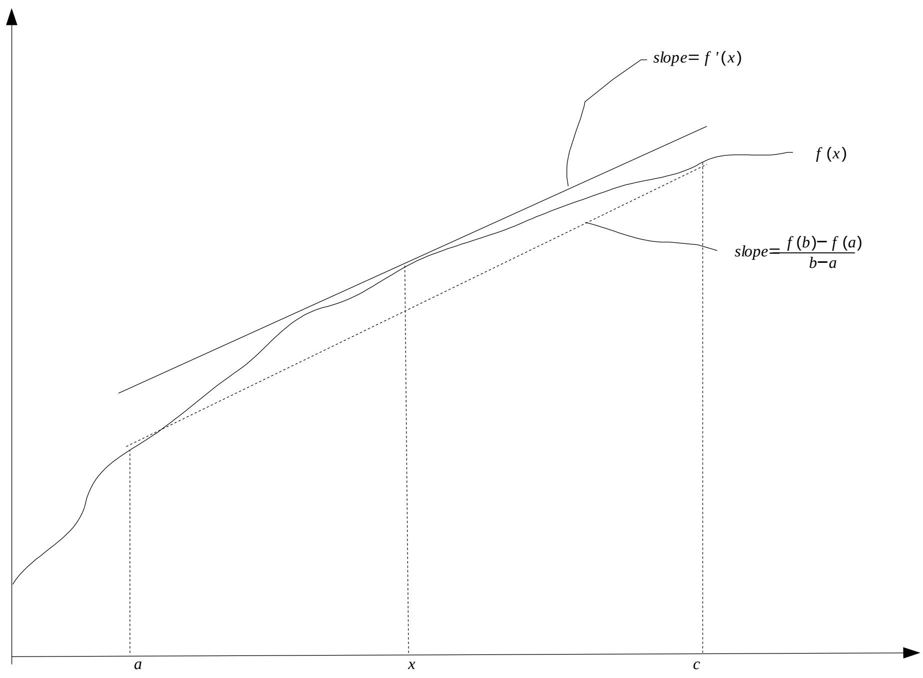

Figure 1.4: The Mean Value Theorem

The following generalizatioin of the Mean Value Theorem is known as the Taylor's Theorem. It may be regarded as showing that a many-times differentiable function can be approximated by a polynomial. The notation $f^{(k)}(z)$ is used in the statement of Taylor's Theorem to denote the k-th derivative of f evaluated at the point z. When $k = 0$. $f^{(k)}(x)$ should be interpreted simply as $f(x)$.

Theorem 1.5.6 ***Taylor's Theorem*** *Let $f : D \to \mathbb{R}$ be a C^m function, where D is an open interval in $\mathbb{R}$, and $m \geq 0$ is a non-negative integer. Suppose also that $f^{(m+1)}(z)$ exists for every point $z \in D$. Then, for any $x, y \in D$, there is $z \in (x, y)$ such that*

$$f(y) = \sum_{k=0}^{m} \left(\frac{f^{(k)}(x)(y - x)^k}{k} \right) + \frac{f^{(m+1)}(z)(y - x)^{m+1}}{(m+1)!}.$$

Proof. See Rudin (1976, Theorem 5.15, p.110) ■

Each of the results you have stated in this subsection, with the obvious exception of the Intermediate Value Theorem for the Derivative, also has an n-dimensional version. These versions you will state here, deriving their proofs as consequences of the corresponding result in $\mathbb{R}$.

Theorem 1.5.7 ***(The Intermediate Value Theorem in $\mathbb{R}^n$)*** *Let $D \subset \mathbb{R}^n$ be a convex set, and let $f : D \to \mathbb{R}$ be continuous on D. Suppose that a and b are points in D such that $f(a) < f(b)$. Then for any c such that $f(a) < c < f(b)$, there is $\hat{\lambda} \in (0, 1)$ such that $f((1 - \hat{\lambda})a + \hat{\lambda}b) = c$.*

Proof. You could derive this result as a consequence of the intermediate Value Theorem in $\mathbb{R}$. Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(\lambda) = \boldsymbol{f}((1 - \lambda)a + \lambda b)$, $\lambda \in [0, 1]$. Since $\boldsymbol{f}$ is a continuous function, g is evidently continuuous on $[0, 1]$. Moreover, $g(0) = \boldsymbol{f}(a)$ and $g(1) = \boldsymbol{f}(b)$, so $g(0) < c < g(1)$. By the Intermediate Value Theorem in $\mathbb{R}$, there exists $\hat{\lambda} \in (0, 1)$ such that $g(\hat{\lambda}) = c$. Since $g(\hat{\lambda}) = \boldsymbol{f}((1 - \hat{\lambda})a + \hat{\lambda}b)$, you are done with the proof. ■

An n-dimensional version of the Mean Value Theorem is similarly established:

Theorem 1.5.8 ***(The Mean Value Theorem in*** $\mathbb{R}^n$***)*** *Let $D \subset \mathbb{R}^n$ be open and convex, and let $\boldsymbol{f} : S \to \mathbb{R}$ be a function that is differentiable everywhere on D. Then, for any $a, b \in D$, there is $\hat{\lambda} \in (0, 1)$ such that*

$$\boldsymbol{f}(b) - \boldsymbol{f}(a) = D\boldsymbol{f}((1 - \hat{\lambda})a + \hat{\lambda}b) \cdot (b - a).$$

Proof. For notational ease, let $z(\lambda) = (1 - \lambda)a + \lambda b$. Define $g : [0, 1] \to \mathbb{R}$ by $g(\lambda) = \boldsymbol{f}(z(\lambda))$ for $\lambda \in [0, 1]$. Note that $g(0) = \boldsymbol{f}(a)$ and $g(1) = \boldsymbol{f}(b)$. Since $\boldsymbol{f}$ is everywhere differentiable by hypothesis, it follows that g is differentiable at all $\lambda \in [0, 1]$, and in fact, $g'(\lambda) = D\boldsymbol{f}(z(\lambda)) \cdot (b - a)$. By the Mean Value Theorem for functions of one variable, therefore, there is $\lambda' \in (0, 1)$ such that

$$g(1) - g(0) = g'(\lambda')(1 - 0) = g'(\lambda').$$

Substituting for g in terms of $\boldsymbol{f}$, this is precisely the statement that

$$\boldsymbol{f}(b) - \boldsymbol{f}(a) = D\boldsymbol{f}(z(\lambda')) \cdot (b - a).$$

You have proved the theorem. ■

Finally, is the Taylor's Theorem in $\mathbb{R}^n$. A complete statement of this result requires some new notation, and is also irrelevant for the remainder of this book. So you are confined to stating two special cases that are useful for your purposes.

Theorem 1.5.9 ***(Taylor's Theorem in*** $\mathbb{R}^n$***)*** *Let $\boldsymbol{f} : D \to \mathbb{R}$, where D is an open set in $\mathbb{R}^n$. If $\boldsymbol{f}$ is C^1 on D, then it is the case that for any $x, y \in D$, you have*

$$\boldsymbol{f}(y) = \boldsymbol{f}(x) + D\boldsymbol{f}(x)(y - x) + R_1(x, y),$$

where the remainder term $R_1(x, y)$ has the property that

$$\lim_{y \to x} \left(\frac{R_1(x, y)}{x - y} \right) = 0.$$

If $\boldsymbol{f}$ is C^2, this statement can be strengthened to

$$\boldsymbol{f}(y) = \boldsymbol{f}(x) + D\boldsymbol{f}(x)(y - x) + \frac{1}{2}(y - x)^t D^2\boldsymbol{f}(x)(y - x) + R_2(x, y).$$

where the remainder term $R_2(x, y)$ has the property that

$$\lim_{y \to x} \left(\right.$$

$R_2(x, y) = 0 \; x - y^2$

Proof. Fix any $x \in D$, and define the function $F(\cdot)$ on D by

$$F(y) = f(x) + Df(x) \cdot (y - x).$$

Let $h(y) = f(y) - F(y)$. Since f and F are C^1, so is h. Note that $h(x) = Dh(x) = 0$. The first-part of the theorem will be proved if you show that

$$\frac{h(y)}{y - x} \to 0 \text{ as } y \to x,$$

or, equivalently, if you show that for any $\ > 0$, there is $\delta > 0$ such that

$$y - x \ < \delta \text{ implies } |h(y)| < \ x - y \ .$$

So let $\ > 0$ be given. By the continuity of h and Dh, there is $\delta > 0$ such that

$$|y - x| < \delta \text{ implies} |h(y)| < \text{ and } Dh(y) \ < \ .$$

Fix any y satisfying $|y - x| < \delta$. Define a function g on $[0, 1]$ by

$$g(t) = h[(1 - t)x + ty].$$

Then $g(0) = h(x) = 0$. Moreover, g is C^1 with $g'(t) = Dh[(1 - t)x + ty](y - x)$.

Now note that $|(1 - t)x + ty - x| = t|(y - x)| < \delta$ for all $t \in [0, 1]$, since $|x - y| < \delta$. Therefore, $Dh[(1 - t)x + ty] \ < \ $ for all $t \in [0, 1]$, and it follows that $|g'(t)| \leq \ y - x$ for all $t \in [0, 1]$.

By Taylor's Theorem in R, there is $t^* \in (0, 1)$ such that

$$g(1) = g(0) + g'(t^*)(1 - 0) = g'(t^*).$$

Therefore,

$$|h(y)| = |g(1)| = |g'(t^*)| \leq \ |y - x|.$$

Since y was an arbitrary point satisfying $|y - x| < \delta$, the first part of the theorem is proved.

You can establish the second part analogously. ■

1.5.3 The Inverse and Implicit Function Theorems

Here, you will state two results of much importance especially for "comparative statics" exercises. The second of these results (The Implicit Function Theorem) also plays a central role in proving Lagrange's Theorem on the first-order conditions for equality-constrained optimization problems. Some new terminology is, unfortunately, required first.

Given a function $f : A \to B$, you will say that the function f maps A *onto* B, if for every $b \in B$, there is some $a \in A$ such that $f(a) = b$. You will say that f is a *one-to-one* function if for any $b \in B$, there is at most one $a \in A$ such that $f(a) = b$. If $f : A \to B$ is both one-to-one and onto, then it is easy to see that there is a (*unique*) function $g : B \to A$ such that $f(g(b)) = b$ for all $b \in B$. (Note that you also have $g(f(a)) = a$ for all $a \in A$.) The function g is called the *inverse function* of ***f***.

Theorem 1.5.10 ***(Inverse Function Theorem)*** *Let $\boldsymbol{f} : S \to R^n$ be a C^1 function, where $S \subset R^n$ is open. Suppose there is a point $y \in S$ such that $n \times n$ matrix $D\boldsymbol{f}(y)$ is invertible. Let $x = \boldsymbol{f}(y)$. Then:*

1. *There are open sets U and V in R^n such that $x \in U$, $y \in V$, $\boldsymbol{f}$ is one-to-one on V, and $\boldsymbol{f}(V) = U$.*
2. *The inverse function $g : U \to V$ of $\boldsymbol{f}$ is C^1 function on U, whose derivative at any point $\hat{x} \in U$ satisfies*

$$Dg(\hat{x}) = (D\boldsymbol{f}(\hat{y}))^{-1}, \quad \text{where} \quad \boldsymbol{f}(\hat{y}) = x$$

Proof. See Rudin (1976, Theorem 9.24, p.221). ■

Turning to the Implicit Function Theorem, the question this result addresses may be motivated by a simple example. Let $S = R^2_{++}$, and let $\boldsymbol{f} : S \to R$ be defined by $\boldsymbol{f}(x, y) = xy$. Pick any point $(\bar{x}, \bar{y}) \in S$, and consider the "level set"

$$C(\bar{x}, \bar{y}) = \{(x, y) \in S | \boldsymbol{f}(x, y) = \boldsymbol{f}(\bar{x}, \bar{y})\}.$$

If you now define the function $h : R_{++} \to R$ by $h(y) = \boldsymbol{f}(\bar{x}, \bar{y})/y$, you have

$$\boldsymbol{f}(h(y), y) \equiv \boldsymbol{f}(\bar{x}, \bar{y})m \quad y \in R_{++}.$$

Thus, the values of the x-variable on the level set $C(\bar{x}, \bar{y})$ can be represented explicitly in terms of the values of the y-variable on this set, through the function h.

In general, an exact form for the original function f may not be specified-for instance, you may only know that $\boldsymbol{f}$ is an increasing C^1 function on R^2-so you may not be able to solve for h explicitly. The question arises whether at least an *implicit* representation of the function h would exist in such a case.

The Implicit Function Theorem studies this problem in a general setting. That is, it looks at sets of functions $\boldsymbol{f}$ from $S \subset R^m$ to R^k, where $m > k$, and asks when the values of some of the variable in the domain can be represented in terms of the others, on a given level set. Under very general conditions, it proves that at least a *local* representatioin is possible.

The statement of the theorem requires a little more notation. Given integers $m \geq 1$ and $n \geq 1$, let a typical point in R^{m+n} be denoted by (x, y), where $x \in R^m$ and $y \in R^n$. For a C^1 function F mapping some subset of R^{m+n} into R^n, let $DF_y(x, y)$ denote that portion of the derivative matrix $DF(x, y)$ corresponding to the last n variables. Note that $DF_y(x, y)$ is an $n \times n$ matrix. $DF_x(x, y)$ is defined similarly.

Theorem 1.5.11 ***Implicit Function Theorem*** *Let $F : S \subset R^{m+n} \to R^n$ be a C^1 function, where S is open. Let (x^*, y^*) be a point in S such that $DF_y(x^*, y^*)$ is invertible, and let $F(x^*, y^*) = c$. Then, there is a neighbourhood $U \subset R^m$ of x^* and a C^1 function $g : U \to R^n$ such that (i) $(x, g(x)) \in S$ for all $x \in U$, (ii) $g(x^*) = y^*$, and (iii) $F(x, g(x)) \equiv c$ for all $x \in U$. The derivative of g at any $x \in U$ may be obtained from the chain rule:*

$$Dg(x) = (DF_y(x, y))^{-1} \cdot DF_x(x, y)$$

Proof. See Rudin (1976, Theorem 9.28, p.224) ■

1.6 Conclusion

In this unit, you have considered some basic concepts as regards to function in R^n, namely, continuity, differentiable and continuous differentiable functions, Partial derivatives and Differentiability, Directional Derivative and Differentiability and Higher Order Derivatives. You also considered Quadratic forms, definite and semidefinite matrices ans some useful results, namely Separation Theorems, The intermediate and Mean value theorem and the inverse and implicit function theorems . All these are great tools which you will use in optimization theory in R^n.

1.7 Summary

Having read through this unit, you are able to

(i) Define Continuous functions, differentiable and continuous differentiable functions, Partial derivatives and Differentiability, Directional derivatives and Differentiability and Higher Order Derivatives.

(ii) Define Quadratic forms and definiteness.

(iii) Identity Definiteness and Semidefiniteness.

(iv) State and Use the Separation Theorems, the Intermediate and Mean Value Theorems, and the Inverse and Implicit Function theorems.

1.8 Tutor Marked Assignments

Exercise 1.8.1

1. Let $\boldsymbol{f} : R^n \to R$ be continuous at a point $p \in R^n$. Assume $\boldsymbol{f}(p) > 0$. Which of the following statements is correct?
   (a) For all open ball $B \subset R^n$ such that $p \in B$, and for all $x \in B$, you have $\boldsymbol{f}(x) > 0$.
   (b) There is an open ball $B \subset R^n$ such that $p \in B$, and for all $x \in B$, you have $\boldsymbol{f}(x) > 0$.
   (c) For all open ball $B \subset R^n$ such that $p \in B$, and there exists $x \in B$, for which $\boldsymbol{f}(x) < 0$.
   (d) There is an open ball $B \subset R^n$ such that $p \in B$, and for all $x \in B$, you have $\boldsymbol{f}(x) < 0$.
2. Suppose $\boldsymbol{f} : R^n \to R$ is continuous function. Then the set

$$\{x \in R^n | \boldsymbol{f}(x) = 0\}$$

is

(a) a closed set

(b) an open set

(c) both open and closed

(d) none of the above.

3. Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find an open set O such that $f^{-1}(O)$ is not open and find a closed set C such that $f^{-1}(C)$ is not closed.

4. Give an example of a function $f : R \to R$ which is continuous at exactly two points (say, at 0 and 1), or show that no such function can exists.

5. Show that it is possible for two function $f : R \to R$ and $g : R \to R$ to be continuous, but for their product $f \cdot g$ to be continuous. What about their composition $f \circ g$?

6. Let $f : R \to R$ be a function which satisfies

$$f(x + y) = f(x)f(y) \quad \text{for all } x, y \in R.$$

Show that if f is continuous at $x = 0$, then it is continuous at every point of R. Also show that if f vanishes at a single point of R, then f vanishes at every point of R.

7. Let $f : R_+ \to R$ be defined by

$$f(x) = \begin{cases} 0, & x = 0 \\ x\sin(1/x), & x = 0 \end{cases}$$

Show that f is continuous at 0.

8. Let D be the unit square $[0, 1] \times [0, 1]$ in R^2. For $(s, t) \in D$, let $f(s, t)$ be defined by

$$f(s, 0) = 0, \quad \text{for all } s \in [0, 1],$$

and for $t > 0$,

$$f(s,t) = \begin{cases} \dfrac{2s}{t} & s \in 0, \dfrac{t}{2} \\ 2 - \dfrac{2s}{t} & s \in \left(\dfrac{t}{2}, t\right. \\ 0 & s \in (t, 1]. \end{cases}$$

(Drawing a picture of f for a fixed t will help). Show that f is a *separately continuous* function, i.e., for each fixed value of t, f is continuous as a function of s, and for each fixed value of s, f is continuous in t. Show also that f is not *jointly continuous* in s and t, i.e., show that there exists a point $(s, t) \in D$ and a sequence (s_n, t_n) in D converging to (s, t) such that $\lim_{n\to} f(s_n, t_n) = f(s, t)$.

9. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} x & \text{If } x \text{ is irrational} \\ 1 - x & \text{if } x \text{ is rational} \end{cases}$$

At what point $x \in R$ is f continuous?

(a) $x = 0$

(b) $x = 1$

(c) $x = \frac{1}{2}$

(d) $x = x_0,\ x_0 \in R$

10. Let $f : R^n \to R$ and $g : R \to R$ be continuous functions. Define $h : R^n \to R$ by $h(x) = g[f(x)]$. Show that h is continuous. Is it possible for h to be continuous even if f and g are not?

11. Show that if a function $f : R \to R$ satisfies

$$|f(x) - f(y)| \leq M(|x - y|)^a$$

for some fixed $M > 0$ and $a > 1$, then f is a constant function, i.e., $f(x)$ is identically equal to some real number b at all $x \in R$.

12. Let $f : R^2 \to R$ be defined by $f(0, 0) = 0$, and for $(x, y) = (0, 0)$,

$$f(x, y) = \frac{xy}{x^2 + y^2}.$$

Show that the two-sided directional derivative of f evaluated at $(x, y) = (0, 0)$ exists in *all* directions $h \in R^2$, but that f is not differentiable at $(0, 0)$.

13. Let $f : R^2 \to R$ be defined by $f(0, 0) = 0$ and for $(x, y) = (0, 0)$

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}.$$

Show that the cross-partials $\partial^2 f(x, y)/\partial x \partial y$ and $\partial^2 f(x, y)/\partial y \partial x$ exist at all $(x, y) \in R^2$, but that these partials are not continuous at $(0, 0)$. Show also that

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{\partial^2 f}{\partial y \partial x}(0, 0).$$

14. Show that an $n \times n$ symmetric matrix A is a positive definite matrix if and only if $-A$ is a negative definite matrix. ($-A$ referes to the matrix whose (i, j)-th entry is $-a_{ij}$.)

15. Prove the following statements or provide a counterexample to show it is false: If A is a positive definite matrix, then A^{-1} is a negative definite matrix.

16. Give an example of matrices A and B which are each negative semidefinite but not negative definite, and which are such that $A + B$ is negative definite.

17. Is it possible for a symmetric matrix A to be simultaneously negative semidefinite *and* positive semidefinite? If yes, give an example. If not, provide a proof.

18. Examine the definiteness or semidefiniteness of the following quadratic forms:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

19. Find the hessians $D^2 f$ of each of the following functions. Evaluate the hessians at the specified points, and examine if the hessian is positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite.

   (a) $f : R^2 \to R$, $f(x) = x_1^2 + \sqrt{x_2}$, at $x = (1, 1)$
   (b) $f : R^2 \to R$, $f(x) = (x_1 x_2)^{1/2}$, at an arbitrary point $x \in R^2_{++}$.
   (c) $f : R^2 \to R$, $f(x) = (x_1 x_2)^2$, at an arbitrary point $x \in R^2_{++}$.
   (d) $f : R^3_+ \to R$, $f(x) = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}$, at $x = (2, 2, 2)$
   (e) $f : R^3_+ \to R$, $f(x) = \sqrt{x_1 x_2 x_3}$, at $x = (2, 2, 2)$.
   (f) $f : R^3_+ \to R$, $f(x) = x_1 x_2 + x_2 x_3 + x_3 x_1$, at $x = (1, 1, 1)$.
   (g) $f : R^3_+ \to R$, $f(x) = ax_1 + bx_2 + cx_3$ for some constants $a, b, c \in R$, at $x = (2, 2, 2)$.

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UNIT 2

OPTIMIZATION IN R^N

2.1 Introduction

This unit constitutes the starting point of your investigation into optimization theory. You will first be introduced to the notation that you will use to represent abstract optimization problems and their solutions and afterwards, address the chief question of interest that will be examined over the book.

2.2 Objectives

At the end of this unit, you should be able to;

(i) Define an optimization problem.

(ii) Give the two types of optimization problems.

(iii) identify a set of conditions on f and D under which the existence of solutions of optimization problems is guaranteed.

2.3 Main Content

2.3.1 Optimization problems in R^n

Definition 2.3.1 *An **optimization problem in** R^n, or simply an **optimization problem**, is one when the values of a given function $f : R^n \to R$ are to be maximized or minimized over a given set $D \subset R^n$. The function f is called the **objective function**, and the set D the **contraint set**.*

Notationally, you will represent these problems by

$$\text{Maximize} \boldsymbol{f}(x) \quad \text{subject to } x \in D$$

and

$$\text{Minimze} \boldsymbol{f}(x) \quad \text{subject to } x \in D$$

respectively. Alternatively, and more compactly, you could also write

$$\max\{\boldsymbol{f}(x)|x \in D\},$$

and

$$\min\{\boldsymbol{f}(x)|x \in D\}.$$

Problems of the first sort are termed *maximization problems* and those of the second sort are called *minimization problems*.

Definition 2.3.2 ***(Solution of an Optimization Problem)*** *A* ***solution*** *to the problem* $\max\{\boldsymbol{f}(x)|x \in D\}$ *is a point x in D such that*

$$\boldsymbol{f}(x) \geq \boldsymbol{f}(y) \text{ for all } y \in D$$

You will say that $\boldsymbol{f}$ *attains a maximum on D at x, and also refer to x as a* maximizer *of* $\boldsymbol{f}$ *on D.*

Similarly, a solution to the problem $\min\{\boldsymbol{f}(x)|x \in D\}$ *is a point z in D such that*

$$\boldsymbol{f}(z) \leq \boldsymbol{f}(y) \text{ for all } y \in D.$$

You will say in this case that $\boldsymbol{f}$ *attains a minimum on D at z, and also refer to z as a* minimizer *of* $\boldsymbol{f}$ *on D.*

Definition 2.3.3 ***(Set of Attainable Values)*** *The set of* ***attainable values*** *of* $\boldsymbol{f}$ *on D, denoted* $\boldsymbol{f}(D)$*, is defined by*

$$\boldsymbol{f}(D) = \{w \in \mathrm{R}| \text{ there is } x \in D \text{ such that } \boldsymbol{f}(x) = w\}.$$

You will also refer to $\boldsymbol{f}(D)$ as the *image of D under* $\boldsymbol{f}$. Observe that $\boldsymbol{f}$ attains a maximum on D (at some x) if and only if the set of real numbers $\boldsymbol{f}(D)$ has a well defined maximum, while $\boldsymbol{f}$ attains a minimum on D (at some z) if and only if $\boldsymbol{f}(D)$ has a well-defined minimum. (This is simply a restatement of the definitions).

The following simple examples reveal two important points: first, that in a given maximization problem, a solution may fail to exist (that is, the problem may have no solution at all), and secondly, that even if a solution does exist, it need not necessarily be unique (that is, there could exist more than one solution). Similar statements obviously also hold for minimization problems.

Example 2.3.1 Let $D = R_+$ and $f(x) = x$ for $x \in D$. Then, $f(D) = R_+$ and $\sup f(D) = +\infty$, so the problem $\max\{f(x)|x \in D\}$ has no solution.

Example 2.3.2 Let $D = [0, 1]$ and let $f(x) = x(1 - x)$ for $x \in D$. Then, the problem of maximizing f on D has exactly one solution, namely the point $x = 1/2$.

Example 2.3.3 Let $D = [-1, 1]$ and $f(x) = x^2$ for $x \in D$. The problem of maximizing f on D now has two solutions: $x = -1$ and $x = 1$.

Thus in the sequel, you will not talk of the solution of a given optimization problem, but of a *set* of solutions of the problem, with the understanding that this set could, in general, be empty. The set of all maximizers of f on D will be denoted $\arg\max\{f(x)|x \in D\}$:

$$\arg\max\{f(x)|x \in D\} = \{x \in D|f(x) \geq f(y) \text{ for all } y \in D\}.$$

The set, $\arg\min\{f(x)|x \in D\}$ of minimizers of f on D is defined analogously. This section shall be closed with two elementary, but important, observations, which is stated in form of theorems for ease of future reference. The first shows that every maximization problem may be represented as a minimzation problem, and *vice versa*. The second identifies a transformation of the optimization problem under which the solution set remains unaffected.

Theorem 2.3.1 *Let $-f$ denote the function whose value at any x is $-f(x)$. Then x is a maximum of f on D if and only if x is a minimum of $-f$ on D and z is a minimizer of f on D if and only if z is maximum of $-f$ on D.*

Proof. The point x maximizes f over D if and only if $f(x) \geq f(y)$ for all $y \in D$, while x minimizes $-f$ over D if and only if $-f(x) \leq -f(y)$ for all $y \in D$. Since $f(x) \geq f(y)$ is the same as $-f(x) \leq -f(y)$, the first part of the theorem is proved. The second part of the theorem follows from the first simply by noting that $-(-f) = f$. ■

Theorem 2.3.2 *Let $\phi : R \to R$ be a strictly increasing function, that is, a function such that*

$$x > y \text{ implies } \phi(x) > \phi(y).$$

Then x is a maximum of f on D if and only if x is also a maximum of the composition $\phi \circ f$ on D; and z is a minimum of f on D, if and only if z is also a minimum of $\phi \circ f$ on D.

Remark 2.3.1 *As will be evident from the proof, it suffices that ϕ be a strictly increasing function on just the set $f(D)$, i.e., that ϕ only satisfy $\phi(z_1) > \phi(z_2)$ for all $z_1, z_2 \in f(D)$ with $z_1 > z_2$.*

Proof. You are dealing with the maximization problem here; the minimization problem is easily deduced using Theorem 2.3.1. Suppose first that x maximizes f over D. Pick any $y \in D$. Then $f(x) \geq f(y)$, and since ϕ is strictly increasing, $\phi(f(x)) \geq \phi(f(y))$. Since $y \in D$ was arbitrary, this inequality holds for all $y \in D$, which states precisely that x is a maximum of $\phi \circ f$ on D.

Now suppose that x maximizes $\phi \circ f$ on D, so $\phi(f(x)) \geq \phi(f(y))$ for all $y \in D$. If x did not also maximize f on D, there would exist $y^* \in D$ such that $f(y^*) > f(x)$. Since ϕ is a

strictly increasing function, it follows that $\phi(\boldsymbol{f}(y^*)) > \phi(\boldsymbol{f}(x))$, so x does not maximize $\phi \circ \boldsymbol{f}$ over D, a contradiction, completing the proof. ■

2.3.2 Types of Optimization problem

In general, There are two types of optimization problem, namely;

1. Unconstrained Optimization problem and
2. Constrained optimization problem.

Unconstrained Optimization problem.

An Optimization problem is called ***unconstrained*** if it is of the form

$$\min_{x \in D} \boldsymbol{f}(x)$$

or

$$\text{min(or max)} \quad \boldsymbol{f}(x)$$
$$\text{Subject to:} \quad x \in D$$

where $x = (x_1, \ldots, x_n) \in R^n$, $\boldsymbol{f} : D \subset R^n \to R$, and D is an open set in R^n

Constrained Optimization Problem

An optimization problem is called ***constrained*** if it is of the form

$$\begin{aligned} &\text{min(or max)} && \boldsymbol{f}(x) \\ &\text{Subject to:} && g_i(x) \geq 0 \quad i = 1, \ldots, m \\ &&& h_i(x) = 0, \quad i = 1, \ldots, l \\ &&& x \in D \end{aligned}$$

where $\boldsymbol{f} : D \subset R^n \to R$ is called the *Objective function*, $g_1, \ldots, g_m, h_1, \ldots, h_l : D \subset R^n \to R$ are the constraint functions.

Let $g = (g_1, \ldots, g_m) : R^m \to R^m$, and $h = (h_1, \ldots, h_l) : R^n \to R^l$, then you can rewrite the constrained problem as follows

$$\begin{aligned} \text{min(or max)} \quad & \boldsymbol{f}(x) \\ \text{Subject to:} \quad & g(x) \geq 0 \\ & h(x) = 0 \\ & x \in D \end{aligned}$$

A detail study of each of the above problems is seen in the next two units.

2.3.3 The Objectives of Optimization Theory

Optimization theory has two main objectives.

1. The first is to identify a set of conditions on $\boldsymbol{f}$ and D underwhich the *existence* of solutions to optimization problems is guaranteed.
2. Second objective lies in obtaining a *characterization* of the set of optimal points. Broad categories of questions of interest here include the following:
   (a) The identification of conditions that every solution to an optimization problem *must* satisfy, that is, of conditions that are *necessary* for an optimum point.
   (b) The identification of conditions such that *any* point that meets these conditions is a solution, that is, of conditions that are *sufficient* to identify a point as being optimal.
   (c) The identification of conditions that ensure only a single solution exists to a given optimization problem, that is, of condition that guarantee *uniqueness* of solutions.

2.4 Existence of Solutions: The Weierstrass Theorem

You will begin the study of optimization with the fundamental question of *existence*: under what conditions on the objective function f and the constraint set D are you *guaranteed* that solutions will always exist in optimization problems of the form $\max\{\boldsymbol{f}(x)|x \in D\}$ or $\min\{\boldsymbol{f}(x)|x \in D\}$? Equivalently, under what conditions on $\boldsymbol{f}$ and D is it the case that the set of attainable values $\boldsymbol{f}(D)$ contains it supremum and/or infimum?. The answer to these questions is given in this section. You will be introduced to two main theorems that gaurantees the existence of solution of an optimization problem. But before that, the following definitions are very important.

Definition 2.4.1 *Let* $\boldsymbol{f} : D \subset \mathbb{R}^n \to \mathbb{R}$ *and let* $\{x_n\}$ *be a sequence of elements in D.* $\{x_n\}$ *is called a* ***minimizing sequence*** *of* $\boldsymbol{f}$ *in D if*

$$\lim_{n \to +\infty} \boldsymbol{f}(x_n) = \inf_{x \in D} \boldsymbol{f}(x)$$

*Similarly $\{x_n\}$ would be called a **maximizing sequence** of f in D if*

$$\lim_{n\to+\infty} f(x_n) = \sup_{x\in D} f(x).$$

Proposition 2.4.1 *If D is a non-empty subset of R^n, then there exists a minimizing (resp. maximizing) sequence $\{x_n\}$ of f in D.*

2.4.1 The Weierstrass Theorem

The following result, a powerful theorem credited to the mathematician Karl Weierstrass, is the main result that answers the questions on existence.

Theorem 2.4.1 (The Weierstrass Theorem) *Let $D \subset R^n$ be compact (i.e., closed and bounded), and let $f : D \to R$ be a continuous function on D. Then f attains a maximum and a minimum on D, i.e., there exists points z_1 and z_2 on D such that*

$$f(z_1) \geq f(x) \geq f(z_2), \; x \in D$$

Or you can write;

$$f(z_1) = \max_{x\in D} f(x) \qquad \textit{and} \qquad f(z_2) = \min_{x\in D} f(x)$$

Proof. The theorem is proved for minimization problem, analogous proof for the maximization problem is readily deduced using Theorem 2.3.1. To proceed, Let, $\{x_n\}$ be a minimizing sequence of f in D. Since D is bounded, by Bolzano-Weierstrass theorem, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to some point $z_1 \in R^n$. Since D is closed, you have that $z_1 \in D$. Using the continuity of f at z_1, it follows that

$$\lim_{k\to+\infty} f(x_{n_k}) = f(z_1) \tag{2.1}$$

On the other hand, since $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$, you have

$$\lim_{k\to+\infty} f(x_{n_k}) = \inf_{x\in D} f(x) \tag{2.2}$$

Using (2.1) and 2.2 and the uniqueness of limit, it follows that

$$f(z_1) = \inf_{x\in D} f(x) = \min_{x\in D} f(x)$$

So z_1 is a *global* minimum of f in D. ∎

It is of the utmost importance to realize that the Weierstrass Theorem only provides *sufficient* conditions for the existence of optima. The theorem has nothing to say about what happens if these conditions are not met, and, indeed, in general, nothing can be said, as the following examples illustrate.

Example 2.4.1 Let $D = R$, and $f(x) = x^3$ for all $x \in R$. The f is continuous but D is not compact (it is closed, but not bounded). Since $f(D) = R$, f evidently attains neither a maximum nor a minimum on D.

Example 2.4.2 Let $D = (0, 1)$ and $f(x) = x$ for all $x \in (0, 1)$. Then f is continuous, but D again noncompact (this time it is bounded, but not closed). The set $f(D)$ is the open interval $(0, 1)$, so, once again, f attains neither a maximum nor a minimum on D.

Example 2.4.3 Let $D = [-1, 1]$, and let f be given by

$$f(x) = \begin{cases} 0, & \text{if } x = -1 \text{ or } x = 1 \\ x, & \text{if } -1 < x < 1 \end{cases}$$

Then D is compact but f fails to be continuous at just the two points -1 and 1. In this case, $f(D)$ is the open interval $(-1, 1)$; consequently, f fails to attain either a maximum or a minimum on D.

Example 2.4.4 Let $D = \mathbb{R}_{++}$, and let $f : D \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if x is rational} \\ 0, & \text{otherwise} \end{cases}$$

Then D is not compact (it is neither closed nor bounded), and f is discontinuous at every single point in R (it "chatters" back and forth between the values 0 and 1). Nonetheless, f attains a maximum (at every rational number) and a minimum (at every irrational number).

To restate the point: if the conditions of the Weierstrass Theorem are met, a maximum and a minimum are guaranteed to exist, On the other hand, if one or more of the theorem's conditions fails, maxima and minima may or may not exist, depending on the specific structure of the problem in question.

Next is the second theorem of existence. But before that, here is an important definition and some propostions that will help you to prove it.

Definition 2.4.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real valued function. f is said to be* ***coercive*** *if*

$$\lim_{\|x\|\to+\infty} f(x) = +\infty$$

Examples

(a) Let $f(x, y) = x^2 + y^2 = \|\mathbf{x}\|^2$. Then

$$\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = \lim_{\|\mathbf{x}\|\to\infty} \|\mathbf{x}\|^2 = \infty$$

Thus $f(x, y)$ is coercive

(b) Let $f(x, y) = x^4 + x^4 - 3xy$. Note that

$$f(x, y) = (x^4 + y^4)\left(1 - \frac{3xy}{x^4 + y^4}\right).$$

If $\|\mathbf{x}\|$ is large, then $3xy/(x^4 + y^4)$ is very small. Hence

$$\lim_{\|(x,y)\|\to\infty} f(x, y) = \lim_{\|(x,y)\|\to\infty} (x^4 + y^4)(1 - 0) = +\infty$$

(c) Let $f(x, y, z) = e^{x^2} + e^{y^2} + e^{z^2} - x^{100} - y^{100} - z^{100}$ then because exponential growth is much faster than the growth of any polynomial, it follows that

$$\lim_{\|(x,y,z)\|\to\infty} f(x, y, z) = \infty$$

Thus $f(x, y, z)$ is coercive.

(d) Linear functions on R^2 ar never coercive. Such functions can be expressed as follows:

$$f(x, y) = ax + by + c$$

where either $a = 0$ or $b = 0$. To see that $f(x, y)$ is not coercive, simply observe $f(x, y)$ is constrantly equal to c on the line

$$ax + by = 0.$$

Since this line is unbounded on this line, the function $f(x, y)$ is not coercive.

(e) If $f(x, y, z) = x^4 + y^4 + z^4 - 3xyz - x^2 - y^2 - z^2$, then as

$$(x, y, z) = \overline{x^2 + y^2 + z^2} \to \infty$$

the higher degree terms dominate and force

$$\lim_{\|(x,y,z)\|\to\infty} f(x, y, z) = \infty.$$

Thus $f(x, y, z)$ is coercive. The following example helps us avoid some misleadings.

(f) Let $f(x, y) = x^2 - 2xy + y^2$. Then

(i) for each fixed y_0, you have $\lim_{|x|\to\infty} f(x, y) = \infty$.

(ii) for each fixed x_0, you have $\lim_{|y|\to\infty} = \infty$;

(iii) but $f(x, y)$ is not coercive.

Properties (i) and (ii) above are more or less clear because in each case the quadratic term dominates. For example, in case (i), you have for a fixed y_0.

$$f(x, y_0) = x^2 - xy_0 + y_0^2.$$

This function of x is a parabola that opens upward. Therefore

$$\lim_{|x|\to\infty} f(x, y_0) = \infty.$$

To see that $f(x, y)$ is not coercive, factor to learn

$$f(x, y) = x^2 - 2xy + y^2 = (x - y)^2.$$

Therefore if (x, y) goes to ∞ on the line $y = +x$, you will see that $f(x, y) = (x - x)^2 = 0$ and hence $f(x, y) = 0$ on the unbounded line $y = x$. Therefore,

$$\lim_{\|(x,y)\|\to\infty} f(x, y) = \infty$$

so $f(x, y)$ is not coercive.

The point of this last example is very important. For $\boldsymbol{f}(\mathbf{x})$ to be coercive, it is not sufficient that $\boldsymbol{f}(\mathbf{x}) \to \infty$ as each coordinate tends to ∞. Rather $\boldsymbol{f}(\mathbf{x})$ must become infinite along any path for which $\mathbf{x}$ becomes infinite.

The reason why coercive functions are important in optimization theory is seen in the next theorem stated shortly.

Proposition 2.4.2 *Let D be a nonempty close subset of $\mathbb{R}^n$. If $\boldsymbol{f}$ is coercive and continuous on some open set containing D, then*

1. *the function $\boldsymbol{f}$ is bounded below (resp. bounded above) on D.*
2. *any minimizing (resp. maximizing) sequence of $\boldsymbol{f}$ in D is bounded.*

Proof. The proof is given for minimization problem.

1. Suppose that $\boldsymbol{f}$ is not bounded below on D. Then for all $n \in \mathbb{N}$, there exists $x_n \in D$ such that $\boldsymbol{f}(x_n) < -n$. So you get a sequence $\{x_n\}$ in D satisfying:

$$\boldsymbol{f}(x_n) < -n, \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.3}$$

This sequence must be bounded because of the coercivity of $\boldsymbol{f}$, otherwise it has a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} x_{n_k} = +\infty.$$

Since $\boldsymbol{f}$ is coercive, you have

$$\lim_{k\to+\infty} \boldsymbol{f}(x_{n_k}) = +\infty.$$

But from (2.3), it follows that

$$\lim_{k\to\infty} \boldsymbol{f}(x_{n_k}) = -\infty$$

and this is a contradiction by the uniqueness of limit. Therefore $\{x_n\}$ is bounded. So by Bolzano-Weierstrass, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to some point $\overline{x} \in D$. Using the continuity of $\boldsymbol{f}$ at $\overline{x}$ it follows that

$$\lim_{k\to\infty} \boldsymbol{f}(x_{n_k}) = \boldsymbol{f}(\overline{x}).$$

From (2.3) you get

$$\lim_{k\to\infty} \boldsymbol{f}(x_{n_k}) = -\infty$$

Therefore, by uniqueness of the limit, it follows that

$$\boldsymbol{f}(\overline{x}) = -\infty,$$

a contradiction, so $\boldsymbol{f}$ is bounded below on D and this ends the proof of 1.

2. Let $\{x_n\}$ be a minimizing sequence of f in D, that is
$$\lim_{n\to\infty} f(x_n) = \inf_{x\in D} f(x). \tag{2.4}$$
You have to show that $\{x_n\}$ is bounded. By contradiction assume that $\{x_n\}$ is not bounded, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that
$$\lim_{k\to\infty} \|x_{n_k}\| = +\infty.$$
Since f is coercive, you have
$$\lim_{k\to\infty} f(x_{n_k}) = +\infty.$$
Using (2.4), you have
$$\lim_{k\to\infty} f(x_{n_k}) = \inf_{x\in D} f(x).$$
and this leads to
$$\inf_{x\in D} f(x) = +\infty.$$
This is a contradiction because of the fact that f is bounded below on D.

■

Theorem 2.4.2 *Let D be a nonempty closed subset of R^n (not necessary bounded). Suppose f is continuous on some open set containing D. Then f has a global minimum on D. That is there exists at least one point $\bar{x} \in D$ such that*
$$f(\bar{x}) = \min_{x\in D} f(x)$$

Proof. Let $\{x_n\}$ be a minimizing sequence of f in D. By 2.4.2, $\{x_n\}$ is bounded, so by Bolzano-Weierstrass theorem $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to some point $\bar{x} \in R^n$. Since D is closed you have $\bar{x} \in D$. Using the continuity of f at $\bar{x}$, it follows that
$$\lim_{k\to+\infty} f(x_{n_k}) = f(\bar{x}). \tag{2.5}$$
On the other hand since $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$, you have
$$\lim_{k\to+\infty} f(x_{n_k}) = \inf_{x\in D} f(x). \tag{2.6}$$
Using (2.5), (2.6) and the uniqueness of limit, it follows that
$$f(\bar{x}) = \inf_{x\in D} f(x)$$
So $\bar{x}$ is a global minimum of f in D. ■

2.5 Conclusion

In this unit you studied optimization in R^n. You looked at what a solution to an optimization problem means and consider two main theorems that guaranteed existence of solution of an optimization problem.

2.6 Summary

Having gone through this unit, you now know

(i) A *Typical Optimization Problem* is

$$\text{Minimize(or Maximize) } f(x) \quad \text{Subject to: } x \in D$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ is called the *objective function* and D is called the constraint set.

(ii) Optimization problems are of two types, namely *Constrained and Unconstrained Problems.* It is constrained if the constraint set D is made up of a set of *inequalities* and/or *equations*

(iii) If for example in the problem

$$\min(\text{ or } \max) \quad f(x) \quad \text{subject to } x \in D$$

that f is continuous and D is a bounded and closed subset of $\mathbb{R}^n$, the there exist a solution for the problem. This is the *Weierstrass Existence theorem theorem.*

(iv) A real valued function $f : \mathbb{R}^n \to \mathbb{R}$ is *coercive* if you have

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

(v) *If f is continuous and coercive on a closed set $D \subset \mathbb{R}$ then there exist $\overline{x} \in D$ such that $f(\overline{x}) \leq f(x)$ for all $x \in D$.*

(ii) the existence theorems for solution of an optimization problem.

2.7 Tutor Marked Assignments(TMAs)

Exercise 2.7.1

1. Prove the following statement or provide a counter example. If f is a continuous function on a bounded (but not necessarily closed) set D, then $\sup f(D)$ is finite.

2. Give an example of an optimization problem with an infinite number of solutions.

3. Let $D = [0, 1]$, Describe the set $f(D)$ in each of the following cases, and identify $\sup f(D)$ and $\inf f(D)$. In which cases does f attain its supremum? What about its infimum?

   (a) $f(x) = 1 + x$ for all $x \in D$

   (b) $f(x) = 1$, if $x < 1/2$, and $f(x) = 2x$ otherwise.

   (c) $f(x) = x$, if $x < 1$ and $f(1) = 2$.

(d) $f(0) = 1$, $f(1) = 0$, and $f(x) = 3x$ for $x \in (0, 1)$.

4. Let $D = [0, 1]$. Suppose $f : D \to R$ is *increasing* on D, i.e., for $x, y \in D$ if $x > y$, then $f(x) > f(y)$. [Note that f is assumed to be continuous on D.] If $f(D)$ a compact set? Prove your answer, or provide a counterexample.
5. Find a function $f : R \to R$ and a collection sets $S_k \subset R$, $k = 1, 2, 3, \ldots$ such that f attains a maximum on each S_k, but not on $\bigcap_{n=1} S_k$.
6. Give an example of a function $f : [0, 1] \to R$ such that $f([0, 1])$ is an open set.
7. Give an example of a set $D \subset R$ and a continuous function $f : D \to R$ such that f attains its maximum, but not a minimum, on D.
8. Let $D = [0, 1]$, Let $f : D \to R$ be an increasing function on D, and let $g : D \to R$ be a decreasing function on D. (That is, if $x, y \in D$ with $x > y$ then $f(x) > f(y)$ and $g(x) < g(y)$.) Then, f attains a minimum and a maximum on D (at 0 and 1, respectively), as does g (at 1 and 0, respectively). Does $f + g$ necessarily attain a maximum and minimum on D?
9. Identify the coercive function in the following list:

   (a) On R^3, let
   $$f(x, y, z) = x^3 + y^3 + z^3 - xy$$
   (b) On R^3, let
   $$f(x, y, z) = x^4 + y^4 + z^2 - 3xy - z.$$
   (c) On R^3, let
   $$f(x, y, z) = x^4 + y^4 + z^2 - 7xyz^2$$
   (d) On R^3, let
   $$f(x, y, z) = x^4 + y^4 - 2xy^2.$$
   (e) On R^3, let
   $$f(x, y, z) = \ln(x^2y^2z^2) - x - y - z.$$
   (f) On R^3, let
   $$f(x, y, z) = x^2 + y^2 + z^2 - \sin(xyz).$$

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UNIT 3

UNCONSTRAINED OPTIMIZATION

3.1 Introduction

In the last unit, you defined an ***unconstrained optimization problem*** as follows

$$\min(\text{or } \max)\boldsymbol{f}(x) \quad x \in D$$

where $\boldsymbol{f} : D \subset \mathbb{R}^n \to \mathbb{R}$ is called the *objective* function. In this unit, you shall be dwelling in this kind of problem in detail.

3.2 Objectives

At the end of this unit, you should be able to

(i) Give the definition of the Local, Global and Strict Optima of an optimization problem.

(ii) State and proof and apply the first order optimality condition for unconstrained optimization problems.

(iii) State, and prove the second order necessary and sufficient condition for an optimization problem. And also use it to solve optimization problems.

(iv) Define Convex sets.

(v) Give the definitions of a Convex function and a Concave function.

(vi) Apply convexity to optimization problems.

3.3 Main Content

The notions, definitions and results you will be seeing hence forth is on the minimization problem.

$$\min \boldsymbol{f}(x) \quad x \in D \tag{3.1}$$

Obvious modifications can be made to yield similar results for maximization problem. But for the sake of simplicity, you will always limit your discussion to minimizers while the minor task of interpreting the results for maximization problems by replacing $\boldsymbol{f}(x)$ by $-\boldsymbol{f}(x)$

3.3.1 Gradients and Hessians

Let $\boldsymbol{f} : D \to \mathbb{R}$, where $D \subset \mathbb{R}^n$ is open, $\boldsymbol{f}$ is *differentiable* at $\bar{x} \in D$ if there exists a vector $\nabla \boldsymbol{f}(\bar{x})$ (called the ***gradient*** of $\boldsymbol{f}$ at $\bar{x}$) such that for each $x \in D$

$$\boldsymbol{f}(x) = \boldsymbol{f}(\bar{x}) + \nabla \boldsymbol{f}(\bar{x})^{\mathbf{t}}(x - \bar{x}) + \|x - \bar{x}\| \alpha(\bar{x}, x - \bar{x}) \tag{3.2}$$

and $\lim_{y \to 0} \alpha(\bar{x}, y) = 0$. $\boldsymbol{f}$ is *differentiable on D* if $\boldsymbol{f}$ is differentiable for all $\bar{x} \in D$. The gradient vector is the vector of partial derivatives:

$$\nabla \boldsymbol{f}(x) = \left(\frac{\partial \boldsymbol{f}}{\partial x_1}(\bar{x}), \dots, \frac{\partial \boldsymbol{f}}{\partial x_n}(\bar{x}) \right)^{\mathbf{t}} \tag{3.3}$$

Example 3.3.1 Let $\boldsymbol{f}(x) = 3x_1^2 x_2^3 + x_2^2 x_3^3$. Then

$$\nabla \boldsymbol{f}(x) = \left(6x_1 x_2^3,\ 9x_1^2 x_2^2 + 2x_2 x_3^3,\ 3x_2^2 x_3^2 \right)^{\mathbf{t}}$$

The *directional derivative* of $\boldsymbol{f}$ at $\bar{\mathbf{x}}$ in the direction $d \in \mathbb{R}^n$ is given by

$$\lim_{\lambda \to 0} \frac{\boldsymbol{f}(\bar{x} + \lambda d) - \boldsymbol{f}(\bar{x})}{\lambda} = \nabla \boldsymbol{f}(\bar{x})^{\mathbf{t}} d \tag{3.4}$$

The function $\boldsymbol{f}$ is *twice differentiable at* $\bar{x} \in D$ if there exists a vector $\nabla \boldsymbol{f}(\bar{x})$ and an $n \times n$ symmetric matrix $H\boldsymbol{f}(\bar{x})$ (called the ***Hessian*** of $\boldsymbol{f}$ at $\bar{x}$) such that for each $x \in D$

$$\boldsymbol{f}(x) = \boldsymbol{f}(\bar{x}) + \nabla \boldsymbol{f}(\bar{x})^{\mathbf{t}}(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^{\mathbf{t}} H\boldsymbol{f}(\bar{x})(x - \bar{x}) + \|x - \bar{x}\|^2 \alpha(\bar{x}, x - \bar{x}), \tag{3.5}$$

and $\lim_{y \to 0} \alpha(\bar{x}, y) = 0$. $\boldsymbol{f}$ is *twice differentiable* on D if and only if $\boldsymbol{f}$ is twice differentiable for all $\bar{x} \in D$. The Hessian is a matrix of second partial derivatives:

$$Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix} \tag{3.6}$$

Example 3.3.2 *Continuing Example 1, you have*

$$Hf(\bar{x}) = \begin{pmatrix} 6x_2^3 & 18x_1x_2^2 & 0 \\ 18x_1x_2^2 & 18x_1^2x_2 + 2x_3^3 & 6x_2x_3^2 \\ 0 & 6x_2x_3^2 & 6x_2^2x_3 \end{pmatrix}$$

3.3.2 Local, Global and Strict Optima

Definition 3.3.1 *Suppose that $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is a real-valued function defined on a subset D of $\mathbb{R}^n$. A point $\bar{x}$ in D is:*

(a) *a **global minimizer** for f on D if $f(\bar{x}) \leq f(x)$ for all $x \in D$;*

(b) *a **strict global minimizer** for f on D if $f(\bar{x}) < f(x)$ for all $x \in D$ such that $x = \bar{x}$;*

(c) *a **local minimizer** for f on D if there is a positive number δ such that $f(\bar{x}) \leq f(x)$ for all $x \in D$ for which $x \in B(\bar{x}, \delta)$;*

(d) *a **strict local minimizer** for f if there is a positive number δ such that $f(\bar{x}) < f(x)$ for all $x \in D$ for which $x \in B(\bar{x}, \delta)$ and $x = \bar{x}$;*

(e) *a **critical point** for f if f is differentiable at $\bar{x}$ and*

$$\nabla f(\bar{x}) = 0.$$

3.3.3 Optimality Conditions For Unconstrained Problems

Before stating the first order optimality condition for the unconstrained problem, the following definition and theorem is needful.

Definition 3.3.2 ***(Descent Direction)*** *The direction $\bar{d}$ is called a descent direction of f at $x = \bar{x}$ if*

$$f(\bar{x} + \bar{d}) < f(\bar{x}) \text{ for all } > 0 \text{ and sufficiently small}$$

A *necessary condition* for local optimality is a statement of the form: "If $\bar{x}$ is a local minimum of (3.1), then $\bar{x}$ must satisfy..." Such a condition will help you to identify all candidates for local optima.

Theorem 3.3.1 *Suppose that $\boldsymbol{f}$ is differentiable at $\bar{x}$. If there is a vector d such that $\nabla \boldsymbol{f}(\bar{x})^{\mathbf{t}}d < 0$, then for all $\lambda > 0$ and sufficiently small, $\boldsymbol{f}(\bar{x}+\lambda d) < \boldsymbol{f}(\bar{x})$, and hence d is a descent direction of $\boldsymbol{f}$ at $\bar{x}$.*

Proof. Suppose there is a vector $d \in \mathbb{R}^n$ such that $\nabla \boldsymbol{f}(\bar{x})^{\mathbf{t}}d < 0$. Since $\boldsymbol{f}$ is differentiable at $\bar{x}$, you have

$$\boldsymbol{f}(\bar{x}+\lambda d) = \boldsymbol{f}(\bar{x}) + \lambda \nabla \boldsymbol{f}(\bar{x})^{\mathbf{t}}d + \lambda \ \ d \ \ \alpha(x, \lambda d).$$

where $\alpha(\bar{x}, \lambda d) \to 0$ as $\lambda \to 0$. Rearranging, you have

$$\frac{\boldsymbol{f}(\bar{x}+\lambda d) - \boldsymbol{f}(\bar{x})}{\lambda} = \nabla \boldsymbol{f}(\bar{x})^{\mathbf{t}}d + \ \ d \ \ \alpha(\bar{x}, \lambda \bar{d}).$$

Since $\nabla \boldsymbol{f}(\bar{x})^{\mathbf{t}}d < 0$ and $\alpha(\bar{x}, \lambda d) \to 0$ as $\lambda \to 0$, $\boldsymbol{f}(\bar{x}+\lambda d) - \boldsymbol{f}(\bar{x}) < 0$ for all $\lambda > 0$ sufficiently small. Thus $\boldsymbol{f}(\bar{x}+\lambda d) < \boldsymbol{f}(\bar{x})$ for all $\lambda > 0$ sufficiently small. ■

Corollary 3.3.1 ***(First Order necessary Optimality condition)*** *Suppose $\boldsymbol{f}$ is differentiable at $\bar{x}$. If $\bar{x}$ is a local minimum then $\nabla \boldsymbol{f}(\bar{x}) = 0$*

Proof. Suppose for contradiction that $\nabla \boldsymbol{f}(\bar{x}) = 0$, then $d = -\nabla \boldsymbol{f}(\bar{x})$ would be a descent direction, whereby $\bar{x}$ would not be a local minimum. Hence, you must have $\nabla \boldsymbol{f}(\bar{x}) = 0$ ■

The above corollary is a *first order neccessary optimality condition* for an unconstrained problem. The following theorem is *second order necessary optimality condition.*

Theorem 3.3.2 ***(Second Order necessary Optimality Condition)*** *Suppose that $\boldsymbol{f}$ is twice continuously differentiable at $\bar{x} \in D$. If $\bar{x}$ is a local minimum, then $\nabla \boldsymbol{f}(\bar{x}) = 0$ and $H\boldsymbol{f}(\bar{x})$ is positive semidefinite.*

Proof. From the first order necessary condition, $\nabla \boldsymbol{f}(\bar{x}) = 0$. Suppose $H\boldsymbol{f}(\bar{x})$ is not positive semidefinite. Then there exists d such that $d^{\mathbf{t}}H\boldsymbol{f}(\bar{x})d < 0$ you have:

$$\begin{aligned}\boldsymbol{f}(\bar{x}+\lambda d) &= \boldsymbol{f}(\bar{x}) + \lambda \nabla \boldsymbol{f}(\bar{x})^{\mathbf{t}}d + \tfrac{1}{2}\lambda^2 d^{\mathbf{t}}H\boldsymbol{f}(\bar{x})d + \lambda^2 \ \ d \ \ ^2\alpha(\bar{x}, \lambda d) \\ &= \boldsymbol{f}(\bar{x}) + \tfrac{1}{2}\lambda^2 d^{\mathbf{t}}H\boldsymbol{f}(\bar{x})d + \lambda^2 \ \ d \ \ ^2\alpha(\bar{x}, \lambda d).\end{aligned}$$

where $\alpha(\bar{x}, \lambda d) \to 0$ as $\lambda \to 0$. Rearranging, gives you

$$\frac{\boldsymbol{f}(\bar{x}+\lambda d) - \boldsymbol{f}(\bar{x})}{\lambda^2} = \frac{1}{2}d^{\mathbf{t}}H\boldsymbol{f}(\bar{x})d + \ \ d \ \ ^2\alpha(\bar{x}, \lambda d).$$

Since $d^{\mathbf{t}}H\boldsymbol{f}(\bar{x})d < 0$ and $\alpha(\bar{x}, \lambda d) \to 0$ as $\lambda \to 0$, $\boldsymbol{f}(\bar{x}+\lambda d) - \boldsymbol{f}(x) < 0$ for all $\lambda > 0$ sufficiently small, yielding the desired condtradition. ■

Example 3.3.3 Let

$$\boldsymbol{f}(x) = \frac{1}{2}x_1^2 + x_1x_2 + 2x_2^2 - 4x_1 - 4x_2 - x_2^3.$$

Then

$$\nabla \boldsymbol{f}(x) = \begin{pmatrix} x_1 + x_2 - 4 \\ x_1 + 4x_2 - 4 - 3x_2^2 \end{pmatrix}$$

and

$$H\boldsymbol{f}(\bar{x}) = \begin{pmatrix} 1 & 1 \\ 1 & 4 - 6x_2 \end{pmatrix}$$

$\nabla \boldsymbol{f}(x) = 0$ has exactly two solutions: $\bar{x} = (4, 0)$ and $\bar{x} = (3, 1)$. But

$$H\boldsymbol{f}(\bar{x}) = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

is indefinite, therefore, the only possible candidate for a local minimum is $\bar{x} = (4, 0)$.

A *sufficient condition* for local optimality is a statement of the form: "If $\bar{x}$ satisfies..., then $\bar{x}$ is a local minimum of 3.1." Such a condition allows you to automatically declare that $\bar{x}$ is indeed a local minimum.

Theorem 3.3.3 ***(Second Order Sufficient Conditon)*** *Suppose that $\boldsymbol{f}$ is twice differentiable at $\bar{x}$. If $\nabla \boldsymbol{f}(\bar{x}) = 0$ and $H\boldsymbol{f}(\bar{x})$ is positive definite, then $\bar{x}$ is a strict local minimum.*

Proof.

$$\boldsymbol{f}(x) = \boldsymbol{f}(\bar{x}) + \frac{1}{2}(x - \bar{x})^{\mathbf{t}} H\boldsymbol{f}(\bar{x})(x - \bar{x}) + \|x - \bar{x}\|^2$$

Suppose that $\bar{x}$ is not a strict local minimum. Then there exists a sequence $\{x_k\}$ which $x_k \to \bar{x}$ as $k \to \infty$ such that $x_k = \bar{x}$ and $\boldsymbol{f}(x_k) \leq \boldsymbol{f}(\bar{x})$ for all k. Define $d_k = \dfrac{x_k - \bar{x}}{\|x_k - \bar{x}\|}$, then

$$\boldsymbol{f}(x_k) = \boldsymbol{f}(\bar{x}) + \|x_k - \bar{x}\|^2 \left(\frac{1}{2} d_k^{\mathbf{t}} H\boldsymbol{f}(\bar{x}) d_k + \alpha(\bar{x}, x_k - \bar{x}) \right).$$

and so

$$\frac{1}{2} d_k^{\mathbf{t}} H\boldsymbol{f}(\bar{x}) d_k + \alpha(\bar{x}, x_k - \bar{x}) = \frac{\boldsymbol{f}(x_k) - \boldsymbol{f}(\bar{x})}{\|x_k - \bar{x}\|^2} \leq 0.$$

Since $\|d_k\| = 1$ for every k, there exists a subsequence of $\{d_k\}$ converging to some point d such that $\|d\| = 1$. Assume without loss of generality that $d_k \to d$, then

$$0 \geq \lim_{k\to\infty} \frac{1}{2} d_k^{\mathbf{t}} H\boldsymbol{f}(\bar{x}) d_k + \alpha(\bar{x}, x_k - \bar{x}) = \frac{1}{2} d^{\mathbf{t}} H\boldsymbol{f}(\bar{x}) d,$$

which is a contradiction of the positive definiteness of $H\boldsymbol{f}(\bar{x})$. ■

Remark 3.3.1 *Note that*

- *If $\nabla \boldsymbol{f}(\bar{x}) = 0$ and $H\boldsymbol{f}(\bar{x})$ is negative definite, then $\bar{x}$ is a local maximum.*

- *If* $\nabla \boldsymbol{f}(\overline{x}) = 0$ *and* $H\boldsymbol{f}(\overline{x})$ *is positive* semidefinite, *you cannot be sure if* $\overline{x}$ *is a local minimum.*

Example 3.3.4 Continuing Example 3.3.3, by computing you have

$$H\boldsymbol{f}(\overline{x}) = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

is positive definite. To see this, note that for any $d = (d_1, d_2)$, you have

$$d^{\mathbf{t}} H\boldsymbol{f}(\overline{x}) d = d_1^2 + 2d_1 d_2 + 4d_2^2 = (d_1 + d_2)^2 + 3d_2^2 > 0 \text{ for all } d = 0$$

Therefore, $\overline{x}$ satifies the sufficient conditions to be a local minimum, and so $\overline{x}$ is a local minimum.

Example 3.3.5 Let

$$\boldsymbol{f}(x) = x_1^3 + x_2^2.$$

Then

$$\nabla \boldsymbol{f}(x) = \begin{pmatrix} 3x_1^2 \\ 2x_2 \end{pmatrix}$$

and

$$H\boldsymbol{f}(x) = \begin{pmatrix} 6x_1 & 0 \\ 0 & 2 \end{pmatrix}$$

At $\overline{x} = (0, 0)$, you have $\nabla \boldsymbol{f}(\overline{x}) = 0$ and

$$H\boldsymbol{f}(\overline{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

is positive semi-definite, but $\overline{x}$ is not a local minimum, since $\boldsymbol{f}(- \ , 0) = - \ ^3 < 0 = \boldsymbol{f}(0, 0) = \boldsymbol{f}(x)$ for all $\ > 0$.

Example 3.3.6 Let

$$\boldsymbol{f}(x) = x_1^4 + x_2^2.$$

Then

$$\nabla \boldsymbol{f}(x) = \begin{pmatrix} 4x_1^3 \\ 2x_2^2 \end{pmatrix}$$

and

$$H\boldsymbol{f}(x) = \begin{pmatrix} 12x_1^2 & 0 \\ 0 & 2 \end{pmatrix}$$

At $\bar{x} = (0, 0)$, you have $\nabla \boldsymbol{f}(\mathbf{x}) = 0$ and

$$Hf(\bar{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

is positive semidefinite. Futhermore, $\bar{x}$ is a local minimum, since for all x you have $\boldsymbol{f}(x) \geq 0 = \boldsymbol{f}(0, 0) = \boldsymbol{f}(\bar{x})$.

FURTHER EXAMPLES

The following examples apply to problems on global minimization whose results are stated in the following theorem.

Theorem 3.3.4 *Suppose that $\bar{x}$ is a critical point of $\boldsymbol{f}$ (i.e., $\nabla \boldsymbol{f}(x) = 0$) is a critical point of a function $\boldsymbol{f}$ with continuous first and second partial derivatives on $\mathbb{R}^n$ and that $H\boldsymbol{f}(x)$ is the Hessian matrix of $\boldsymbol{f}$. Then $\bar{x}$ is:*

(a) global minimizer for $\boldsymbol{f}$ if $H\boldsymbol{f}(x)$ is positive semidefinite on $\mathbb{R}^n$; (b) a strict global minimizer of $\boldsymbol{f}$ if $H\boldsymbol{f}(x)$ is positive definite on $\mathbb{R}^n$; (c) a global maximizer for $\boldsymbol{f}$ if $H\boldsymbol{f}(x)$ is negative semidefinite on $\mathbb{R}^n$; (d) a strict global maximizer for $\boldsymbol{f}$ if $H\boldsymbol{f}(x)$ is negative definite on $\mathbb{R}^n$.

Here are four examples that summarizes the above result you now know.

Example 3.3.7

(a) Minimize the function $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\boldsymbol{f}(x) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_1x_3, \text{ for all } x = (x_1, x_2, x_3) \in \mathbb{R}^3$$

☞ ***Solution.*** The critical points of $\boldsymbol{f}$ are the solutions of the system

$$\nabla \boldsymbol{f}(x) = \begin{pmatrix} 2x_1 - x_2 - x_3 \\ -x_1 + 2x_2 + x_3 \\ -x_1 + x_2 + 2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The one and only solution to the system is $x_1 = 0, x_2 = 0, x_3 = 0$ The Hessian of $\boldsymbol{f}(x)$ is a constant marix

$$H\boldsymbol{f}(x) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

Note that $\Delta_1 = 2$, $\Delta_2 = 3$, $\Delta_3 = 4$ so $H\boldsymbol{f}(x)$ is positive definite everywhere on everywhere on $\mathbb{R}^3$. It follows for Theorem 3.3.4 that the critical point $(0, 0, 0)$ is a strict global minimizer for $\boldsymbol{f}$.

Since $\boldsymbol{f}$ is defined and has continuous first partial derivatives everywhere on $\mathbb{R}^3$ and since $(0, 0, 0)$ is the only critical point of $\boldsymbol{f}$, it follows for Corollary 3.3.1 that $\boldsymbol{f}$ has no other minimizers or maximizers. ✍

(b) Find the global minimizer of

$$\boldsymbol{f}(x, y, z) = e^{x-y} + e^{y-x} + e^{x^2} + z^2.$$

☞ ***Solution.*** To this end, compute

$$\nabla \boldsymbol{f}(x, y, z) = \begin{pmatrix} e^{x-y} - e^{y-x} + 2xe^{x^2} \\ -e^{x-y} + e^{y-x} \\ 2z \end{pmatrix}$$

and

$$H\boldsymbol{f}(x, y, z) = \begin{pmatrix} e^{x-y} + e^{y-x} + 4x^2e^{x^2} + 2e^{x^2} & -e^{x-y} - e^{y-x} & 0 \\ -e^{x-y} - e^{y-x} & e^{x-y} + e^{y-x} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Clearly, $\Delta_1 > 0$ for all x, y, z because all the terms of it are positive. Also

$$\begin{aligned} \Delta_2 &= (e^{x-y} + e^{y-x})^2 + (e^{x-y} + e^{y-x})(4x^2e^{x^2} + 2e^{x^2}) - (e^{x-y} + e^{y-x})^2 \\ &= (e^{x-y} + e^{y-x})(4x^2e^{x^2} + 2e^{x^2}) > 0. \end{aligned}$$

because both factors are always positive. Finally, $\Delta_3 = 2\Delta_2 > 0$. Hence $H\boldsymbol{f}(x, y, z)$ is positive definite at all points. Therefore by Theorem 3.3.4 $\boldsymbol{f}$ is strictly globally minimized by any critical point $(\bar{x}, \bar{y}, \bar{z})$. To find $(\bar{x}, \bar{y}, \bar{z})$, solve

$$0 = \nabla \boldsymbol{f}(\bar{x}, \bar{y}, \bar{z}) = \begin{pmatrix} e^{\bar{x}-\bar{y}} - e^{\bar{y}-\bar{x}} + 2\bar{x}e^{\bar{x}^2} \\ -e^{\bar{x}-\bar{y}} + e^{\bar{y}-\bar{x}} \\ 2\bar{z} \end{pmatrix}$$

This leads to $\bar{z} = 0$, $e^{\bar{x}-\bar{y}} = e^{\bar{y}-\bar{x}}$, hence $2\bar{x}e^{\bar{x}^2} = 0$. Accordingly, $\bar{x} - \bar{y} = \bar{y} - \bar{x}$; that is, $\bar{x} = \bar{y}$ and $\bar{x} = 0$. Therefore $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 0)$ is the strict global minimizer of $\boldsymbol{f}$. ✍

(c) Find the global minimizers of

$$\boldsymbol{f}(x, y) = e^{x-y} + e^{y-x}$$

☞ ***Solution.*** To this end, compute

$$\nabla \boldsymbol{f}(x, y) = \begin{bmatrix} e^{x-y} - e^{y-x} \\ -e^{x-y} + e^{y-x} \end{bmatrix}$$

and

$$H\boldsymbol{f}(x, y) = \begin{bmatrix} e^{x-y} + e^{y-x} & -e^{x-y} - e^{y-x} \\ -e^{x-y} - e^{y-x} & e^{x-y} + e^{y-x} \end{bmatrix}$$

Since $e^{x-y} + e^{y-x} > 0$ for all x, y and $\det H\boldsymbol{f}(x, y) = 0$, then, by the Hessian $H\boldsymbol{f}(x, y)$ is positive semidefinite for all x, y. Therefore, by 3.3.4, $\boldsymbol{f}(x, y)$ is minimized at any critical point $(\bar{x}, \bar{y})$ of $\boldsymbol{f}(x, y)$. To find $(\bar{x}, \bar{y})$, solve

$$0 = \nabla \boldsymbol{f}(\bar{x}, \bar{y}) = \begin{bmatrix} e^{\bar{x}-\bar{y}} - e^{\bar{y}-\bar{x}} \\ -e^{\bar{x}-\bar{y}} + e^{\bar{y}-\bar{x}} \end{bmatrix}$$

This gives

$$e^{\bar{x}-\bar{y}} = e^{\bar{y}-\bar{x}}$$

or

$$\bar{x} - \bar{y} = \bar{y} - \bar{x};$$

that is,

$$2\bar{x} = 2\bar{y}.$$

This shows that all points of the line $y = x$ are global minimizers of $\boldsymbol{f}(x, y)$. ✍

(d) Find the global minimizers of

$$\boldsymbol{f}(x, y) = e^{x-y} + e^{x+y}$$

☞ ***Solution.*** In this case,

$$\nabla \boldsymbol{f}(x, y) = \begin{bmatrix} e^{x-y} + e^{x+y} \\ -e^{x-y} + e^{x+y} \end{bmatrix}$$

$$H\boldsymbol{f}(x, y) = \begin{bmatrix} e^{x-y} + e^{x+y} & -e^{x-y} + e^{x+y} \\ -e^{x-y} + e^{x+y} & e^{x-y} + e^{x+y} \end{bmatrix}.$$

Since $e^{x-y} + e^{x+y} > 0$ for all x, y and $\det H\boldsymbol{f}(x, y) > 0$, then $H\boldsymbol{f}(x, y)$ is positive definite for all x, y. Therefore, by Theorem 3.3.4, $\boldsymbol{f}(x, y)$ is minimized at any critical point $(\bar{x}, \bar{y})$. To find $(\bar{x}, \bar{y})$, write

$$0 = \nabla \boldsymbol{f}(\bar{x}, \bar{y}) = \begin{bmatrix} e^{\bar{x}-\bar{y}} + e^{\bar{x}+\bar{y}} \\ e^{\bar{x}-\bar{y}} + e^{\bar{x}+\bar{y}} \end{bmatrix}$$

Thus

$$e^{\bar{x}-\bar{y}} + e^{\bar{x}+\bar{y}} = 0$$

and

$$-e^{\bar{x}-\bar{y}} + e^{\bar{x}+\bar{y}} = 0$$

But $e^{\bar{x}-\bar{y}} > 0$ and $e^{\bar{x}+\bar{y}} > 0$ for all $\bar{x}, \bar{y}$. Therefore the equality $e^{\bar{x}-\bar{y}} + e^{\bar{x}+\bar{y}} = 0$ is impossible. Thus $f(x, y)$ has no critical points and hence $f(x, y)$ has no global minimizers. ✍

3.3.4 Coercive functions and Global Minimizers

You could remember that in the preceeding unit, you said that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *coercive* if

$$\lim_{\|x\| \to +\infty} f(x) = +\infty$$

and you also stated a very important result on existence of global minimizers for coercive functions in Theorem 2.4.2 which says that if D is a closed set and $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is continuous and coercive on some open set containing D, then there exists a global minimizer of f in D.

In addition to this theorem, is that if the first partial derivatives of f exist on all of $\mathbb{R}^n$, then these global minimizers can be found among the critical points of f. Here is an example to illustrate this notion.

Example 3.3.8 Minimize

$$f(x, y) = x^4 - 4xy + y^4$$

on $\mathbb{R}^2$.

☞ ***Solution.*** To this end, compute

$$\nabla f(x, y) = \begin{pmatrix} 4x^3 - 4y \\ -4x + 4y^3 \end{pmatrix}$$

and

$$Hf(x, y) = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

Note that

$$Hf\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$$

which is certainly not positive definite since $\det H\left(\frac{1}{2}, \frac{1}{2}\right) = 9 - 16 < 0$. Therefore the tests from the last section are not applicable. But all is not lost because f is coercive!

To see that f is coercive, note that

$$f(x, y) = x^4 + y^4 \left(1 - \frac{4xy}{x^4 + y^4}\right)$$

As $\|(x, y)\| = \sqrt{x^2 + y^2} \to +\infty$, the term $4xy/(x^4 + y^4) \to 0$. Hence

$$\lim_{\|(x,y)\|\to\infty} f(x, y) = \lim_{\|(x,y)\|\to\infty} (x^4 + y^4)(1 - 0) = +\infty.$$

Thus f is coercive. According to Theorem 2.4.2 f has a global minimizer at one of the critical points. Setting $\nabla f(x, y) = 0$, you get $y = x^3$, and $x = y^3$. Hence $x = x^9$ and $x(x^8 - 1) = 0$. This produces three critical points

$$(0, 0), (1, 1), (-1, -1)$$

Now

$$f(0, 0) = 0, \quad f(1, 1) = -2 \quad f(-1, -1) = -2$$

Therefore $(-1, -1)$ and $(1, 1)$ are both global minimizers of f.

✍

3.4 Convex Sets and Convex Functions

It is necessary at this point that you study convexity briefly because of some of its important considerations in optimization theory. Which include First, *Convex functions occur frequently and naturally in many optimization problems that arise in statistical, economical, or industrial applications.* Second, *convexity often make it unnecessary to test the Hessians of functions for positive definiteness, a test which can be difficult in practice as you have seen in the preceding section.*

You will be introduced to a very basic concept of Convexity and then state some important results which will help you minimize a function.

3.4.1 Convex Sets

Definition 3.4.1 *A set C in $\mathbb{R}^n$ is* **convex** *if for every $x, y \in C$, the line segment joining x and y remains inside C.*

The *line segment* $[x, y]$ joining x and y is defined by

$$[x, y] = \{\lambda x + (1 - \lambda)y : 0 \le \lambda \le 1\}.$$

Therefore, a subset C in $\mathbb{R}^n$ is convex if and only if for every x and y in C and every λ with $0 \le \lambda \le 1$, the vector $\lambda x + (1 - \lambda)y$ is also in

C.

Examples of Convex Sets

(a) Let x and v be vectors in $\mathbb{R}^n$. The line L through x in the direction of v

$$L = \{x + \lambda v, \lambda \in \mathbb{R}\}$$

is convex set in $\mathbb{R}^n$.

(b) Any linear subspace M of $\mathbb{R}^n$ is a convex set since linear subspaces are closed under addition and scalar multiplication.

(c) If $\bar{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then the closed half-spaces

$$F^+ = \{y \in \mathbb{R}^n : \bar{x} \cdot y \geq \alpha\} \quad F^- = \{y \in \mathbb{R}^n : \bar{x} \cdot y \leq \alpha\}$$

determined by $\bar{x}$ and α are all convex sets.

(d) If $\bar{x} \in \mathbb{R}^n$ and $r > 0$ then the ball centered at $\bar{x}$ with radius r

$$B'(x, r) = \{x \in \mathbb{R}^n : \|x - \bar{x}\| \leq r\}$$

is a convex set in $\mathbb{R}^n$.

Theorem 3.4.1 *Let C be a convex subset in $\mathbb{R}^n$. Let $x_1, \ldots, x_m$ be points in C. If $\lambda_1, \ldots, \lambda_m$ are non-negative numbers whose sum is 1 then the conves combination*

$$\lambda x_1 + \cdots + \lambda_m x_m$$

is also in C.

Proof. Assume that the nonempty set C is convex, you have to show that C contains all its convex combinations. You can proceed by induction as follows. Define the property P_n as follows;

$$\mathbf{P}_n : \sum_{i=1}^{n} \lambda_i x_i \in C \text{ for all } x_1, \ldots, x_n \in C, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1$$

1. The property obviously hold for $n = 1$, i.e., (P_1) is fulfilled.
2. Assume that properties $(P_1), \ldots, (P_n)$ holds. Let $x_1, \ldots, x_n, x_{n+1} \in C, \lambda_1 \geq, \lambda_2 \geq 0, \ldots, \lambda_n \geq 0, \lambda_{n+1} \geq 0$ with

$$\sum_{i=1}^{n+1} \lambda_i = 1$$

Of course, if $\lambda_{n+1} = 1$, then

$$\sum_{i=1}^{n+1} \lambda_i x_i = x_{n+1} \in C,$$

because $\lambda_1 = \cdots = \lambda_n = 0$ in this case. And so

$$\sum_{i=1}^{n+1} \lambda_i x_i \in C.$$

Assume that $\lambda_{n+1} = 1$. This allows you to write

$$\begin{aligned}\sum_{i=1}^{n+1}\lambda_i x_i &= \sum_{i=1}^{n}\lambda_i x_i + \lambda_{n+1}x_{n+1}\\ &= (1-\lambda_{n+1})\left(\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}}x_i\right) + \lambda_{n+1}x_{n+1}.\end{aligned}\tag{3.7}$$

You have

$$\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}} = \frac{1}{1-\lambda_{n+1}}\sum_{i=1}^{n}\lambda_i = \frac{1}{1-\lambda_{n+1}}(1-\lambda_{n+1}) = 1, \quad \text{since} \quad \sum_{i=1}^{n+1}\lambda_i = 1$$

and

$$\frac{\lambda_i}{1-\lambda_n} \geq 0 \text{ and } x_1, \ldots, x_n \in C,$$

hence by induction assumption,

$$x' := \sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}}x_i \in C$$

Since $y' := x_{n+1} \in C$ by assumption you get that

$$(1-\lambda_{n+1})x' + \lambda_{n+1}y' \in C \tag{3.8}$$

because $\lambda_{n+1} \in [0, 1]$. Combining (3.7) and (3.8) you can conclude that

$$\sum_{i=1}^{n+1}\lambda_i x_i \in C$$

This completes the proof. ■

The preceeding argument demonstrates that if C contains any convex combination of two of its points, then it must also contain any convex combination of three of its points.

3.4.2 Convex Functions

Definition 3.4.2 Let C be a convex nonempty subset of $\mathbb{R}^n$ and $\boldsymbol{f}$ a real-valued function from C to $\mathbb{R}$. Then

(a) the function $\boldsymbol{f}$ is a ***convex*** function if

$$\boldsymbol{f}(\lambda x + (1-\lambda)y) \leq \lambda\boldsymbol{f}(x) + (1-\lambda)\boldsymbol{f}(y)$$

for all $x, y \in C$, and all λ with $0 \leq \lambda \leq 1$.

(b) the function f is a ***strictly convex*** function if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in C$ with $x = y$ and all λ with $0 < \lambda < 1$.

(c) the function f is ***concave*** function if

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in C$ and for all λ with $0 \leq \lambda \leq 1$.

(d) the function f is a ***strictly concave*** function if

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in C$ with $x = y$ and all λ with $0 < \lambda < 1$

Remark 3.4.1 *Note that f is convex (resp. strictly convex) on a convex set C if and only if $-f$ is a concave (resp. strictly concave) on C. Because of this close connection, all results are formulated in terms of convex functions only. Corresponding results for concave functions will be clear.*

Example 3.4.1

1. Any linear function of n variables is both convex and concave on $\mathbb{R}^n$.
2. The function $f(x) = (a \cdot x)^2$ where a is a fixed vector in $\mathbb{R}^n$ is convex on $\mathbb{R}^n$.

Theorem 3.4.2 *Suppose that f is a convex function defined on a convex subset C of $\mathbb{R}^n$. If $\lambda_1, \ldots, \lambda_m$ are non-negative numbers with sum 1 and if $x_1, \ldots, x_m$ are points of C, then*

$$f\left(\sum_{k=1}^{m} \lambda_k x_k\right) \leq \sum_{k=1}^{m} \lambda_k f(x_k) \tag{3.9}$$

If f is strictly convex on C and if all the λ_k's are positive then equality holds in (3.9) if and only if all the x_k's are equal.

3.4.3 Convexity and Optimization

The results proved in this section link convexity to optimization.

Theorem 3.4.3 *Suppose C is a convex subset of $\mathbb{R}^n$, $f : C \to \mathbb{R}$ is a convex function and $\bar{x}$ is a local minimum of f. Then $\bar{x}$ is also a global minimum of f in C. In addition, if f is a strictly convex function, then $\bar{x}$ is a unique global minimum of f in C.*

Proof. Suppose that $\bar{x}$ is a local minimizer of $\boldsymbol{f}$ in C. Then there exists a positive number $\boldsymbol{r}$ such that

$$\boldsymbol{f}(\bar{x}) \leq \boldsymbol{f}(x), \quad \text{for all} \quad x \in C \cap B(\bar{x}, \boldsymbol{r})$$

Given $x \in C$, you have to show that $\boldsymbol{f}(\bar{x}) \leq \boldsymbol{f}(x)$. To this end, select λ, with $0 < \lambda < 1$ and so small that

$$\bar{x} + \lambda(x - \bar{x}) = \lambda x + (1 - \lambda)\bar{x} \in C \cap B(\bar{x}, \boldsymbol{r})$$

Then

$$\boldsymbol{f}(\bar{x}) \leq \boldsymbol{f}(\bar{x} + \lambda(x - \bar{x})) = \boldsymbol{f}(\lambda x + (1 - \lambda)\bar{x}) \leq \lambda \boldsymbol{f}(x) + (1 - \lambda)\boldsymbol{f}(\bar{x})$$

because $\boldsymbol{f}$ is convex. Now subtract $\boldsymbol{f}(\bar{x})$ from both sides of the preceeding inequality, and divide the result by λ to obtain $0 \leq \boldsymbol{f}(x) - \boldsymbol{f}(\bar{x})$. This establishes that $\bar{x}$ is a global minimum.

Now suppose $\boldsymbol{f}$ is strictly convex. Let x_1 and x_2 be two different minimizers of $\boldsymbol{f}$ and let λ with $0 < \lambda < 1$. Because of the strict convexity of $\boldsymbol{f}$ and the fact that

$$\boldsymbol{f}(x_1) = \boldsymbol{f}(x_2) = \min_{x \in C} \boldsymbol{f}(x)$$

you have

$$\boldsymbol{f}(x_1) \leq \boldsymbol{f}(\lambda x_1 + (1 - \lambda)x_2) < \lambda \boldsymbol{f}(x_1) + (1 - \lambda)\boldsymbol{f}(x_2) = \boldsymbol{f}(x_1)$$

which is a contradiction, therefore, $x_1 = x_2$. ■

Remark 3.4.2

- *If $\boldsymbol{f}$ is a concave function, then a local maximum is a global maximum.*
- *If $\boldsymbol{f}$ is a strictly concave function, then a local maximum is a unique global maximum.*

Theorem 3.4.4 (Gradient Inequality). *Suppose that $\boldsymbol{f}$ has continuous first partial derivatives on some open set containing the convex set C. Then*

1. *The function $\boldsymbol{f}$ is convex if and only if*

$$\boldsymbol{f}(y) \geq \boldsymbol{f}(x) + \nabla \boldsymbol{f}(x)^{\mathbf{t}}(y - x) \text{ for all } x, y \in C \tag{3.10}$$

2. *The function $\boldsymbol{f}$ is strictly convex if and only if*

$$\boldsymbol{f}(y) > \boldsymbol{f}(x) + \nabla \boldsymbol{f}(x)^{\mathbf{t}}(y - x) \text{ for all } x, y \in C \tag{3.11}$$

Proof. The proof of no. 1 is given here. Suppose that $\boldsymbol{f}$ is convex on C. Let $x, y \in C$ and λ with $0 < \lambda < 1$. Then

$$\boldsymbol{f}(x + \lambda(y - x)) = \boldsymbol{f}(\lambda y + (1 - \lambda)x) \leq \lambda \boldsymbol{f}(y) + (1 - \lambda)\boldsymbol{f}(x)$$

so that

$$\frac{f(x+\lambda(y-x))-f(x)}{\lambda} \leq f(y)-f(x).$$

If you let $\lambda \to 0$, you obtain

$\nabla f(x) \cdot (y - x) \leq f(y) - f(x)$

Therefore

$$f(y) \geq f(x) + \nabla f(x)^{\mathbf{t}}(y - x)$$

for all $x, y \in C$.

Conversely, suppose that inequality (3.10) holds for all $x, y \in C$. Let w and z be any two points in C. Let $\lambda \in [0, 1]$, and set $x = \lambda w + (1 - \lambda)z$. Then

$$f(w) \geq f(x) + \nabla f(x)^{\mathbf{t}}(w - x) \quad \text{and} \quad f(z) \geq f(x) + \nabla f(x)^{\mathbf{t}}(z - x)$$

Taking a convex combination of the above inequalities, you obtain

$$\begin{aligned} \lambda f(w) + (1 - \lambda)f(z) &\geq f(x) + \nabla f(x)^{\mathbf{t}}(\lambda(w - x) + (1 - \lambda)(z - x)) \\ &= f(x) + \nabla f(x)^{\mathbf{t}}0 \\ &= f(\lambda w + (1 - \lambda)z), \end{aligned}$$

which shows that f is convex. ■

The following striking result is an immediate consequence of Theorem 3.4.4. It is the most important and useful result in this chapter.

Corollary 3.4.1 *If f is a convex function with continuous first partial derivatives on some open set containing the convex set C, then any critical point of f in C is a global minimizer of f.*

Proof. Suppose that $\overline{x} \in C$ is a critical point of f. Let $x \in C$. Then $\nabla f(\overline{x}) = 0$ and (3.10) imply that

$$f(\overline{x}) = f(\overline{x}) + \nabla f(\overline{x})^{\mathbf{t}}(x - \overline{x}) \leq f(x).$$

Consequently, $\overline{x}$ is a global minimizer of f on C. ■

Although the definitions of convex and strictly convex functions and the gradient inequalities provide useful tools for deriving important information concerning their properties, they are not very useful for recognizing convex and strictly convex functions in concrete examples. For instance, the function $f(x) = x^2$ is certainly convex (even strictly convex) function on $\mathbb{R}^n$, yet it is cumbersome to verify this fact by using definition or the gradient inequality of convex function. The next two theorems will provide you with an effective means for recognizing convex functions in specific examples.

Theorem 3.4.5 *Suppose that f has continuous second partial derivatives on some open convex set C in $\mathbb{R}^n$. Let $Hf(x)$ be the Hessian matrix of f. then f is convex on C if and only if $Hf(x)$ is positive semidefinite for all $x \in C$.*

Proof. Suppose f is convex. Let $\overline{x} \in C$ and d be any direction. Then for $\lambda > 0$ sufficiently small, $\overline{x} + \lambda d \in C$. You have:

$$f(\overline{x} + \lambda d) = f(\overline{x}) + \nabla f(\overline{x})^{\mathbf{t}}(\lambda d) + \frac{1}{2}(\lambda d)^{\mathbf{t}}Hf(x)(\lambda d) + \|\lambda d\|^2\alpha(\overline{x}, \lambda d),$$

where $\alpha(\overline{x}, y) \to 0$ as $y \to 0$. Using the gradient inequality, you obtain

$$\lambda^2 \left(\frac{1}{2} d^{t} H f(\overline{x}) d + d^{2} \alpha(\overline{x}, \lambda d) \right) \geq 0.$$

Dividing by $\lambda^2 > 0$ and letting $\lambda \to 0$, you obtain $d^{\mathbf{t}} H\boldsymbol{f}(\overline{x})d \geq 0$, i.e., $H\boldsymbol{f}(\overline{x})$ is positive semidefinite. This completes the proof of this direction.

Conversely, suppose that $H\boldsymbol{f}(z)$ is postive semidefinite for all $z \in C$. Let $x, y \in C$ be arbitrary. Invoking the second-order version of Taylor's theorem, you have:

$$\boldsymbol{f}(y) = \boldsymbol{f}(x) + \nabla \boldsymbol{f}(x)^{\mathbf{t}}(y - x) + \frac{1}{2}(y - x)^{\mathbf{t}} H\boldsymbol{f}(z)(y - x)$$

for some z which is a convex combination of x and y (and hence $z \in C$). Since $H\boldsymbol{f}(z)$ is positive semidefinite, this means that

$$\boldsymbol{f}(y) \geq \boldsymbol{f}(x) + \nabla \boldsymbol{f}(x)^{\mathbf{t}}(y - x).$$

Therefore the gradient inequality holds, and hence $\boldsymbol{f}$ is convex. ∎

The following example illustrates how Theorem 3.4.5 can be applied to test convexity.

Example 3.4.2 Consider the function $\boldsymbol{f}$ defined on R^3 by

$$\boldsymbol{f}(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + x_3^2 + 2x_2x_3.$$

The Hessian of $\boldsymbol{f}$ is

$$H\boldsymbol{f}(x) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

The principal minors of $H\boldsymbol{f}(x)$ are $\Delta_1 = 4$, $\Delta_2 = 8$, $\Delta_3 = 0$, Which implies that $H\boldsymbol{f}(x)$ is positive semidefinite, and so $\boldsymbol{f}$ is convex by Theorem 3.4.5. Since $H\boldsymbol{f}(x)$ is not positive definite, it is not possible to conclude from Theorem 3.4.5 that $\boldsymbol{f}$ is strictly convex on R^3. As a matter of fact, since

$$\boldsymbol{f}(x_1, x_2, x_3) = 2x_1^2 + (x_2 + x_3)^2,$$

you see that $\boldsymbol{f}(x) = 0$ for all x on the line where $x_1 = 0$ and $x_3 = -x_2$, so $\boldsymbol{f}$ is not strictly convex.

The discussion above shows that many of the results of the preceeding section, are subsumed under the general heading of convex functions. But you must note that verifying that the Hessian is postive semidefinite is sometimes difficult. For instance, the function

$$\boldsymbol{f}(x, y, z) = e^{x^2+y+z} - \ln(x + y) + 3^{z^2}$$

is convex on R^3 but its Hessian is a mess. Fortunately, there are ways other then checking the Hessian to show that a function is convex. The next group of results points in this direction. The following theorem shows that convex functions can be combined in a variety of ways to produce new convex functions.

Theorem 3.4.6

(a) *If $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m$ are convex functions on a convex set C in R^n, then*

$$\boldsymbol{f}(x) = \boldsymbol{f}_1(x) + \cdots + \boldsymbol{f}_m(x)$$

is convex. Moreover, if at least one $\boldsymbol{f}_i(x)$ is strictly convex on C, then the sum $\boldsymbol{f}$ is strictly convex.

(b) *If f is convex (resp. strictly convex) on a convex set C in $\mathbb{R}^n$ and if α is a positive number, then αf is convex (resp. strictly convex) on C.*

(c) *If f is convex (resp. strictly convex) function defined on a convex set C in $\mathbb{R}^n$, and if ϕ is an increasing (resp. strictly increasing) convex function defined on the range of f in $\mathbb{R}$, then the composite function $\phi \circ f$ is convex (resp. strictly convex).*

Proof.

(a) To show that any finite sum of convex function on C is convex on C, it suffices to show that the sum $(f_1 + f_2)$ of two convex functions f_1 and f_2 on C is again convex on C. If, y, z belong to C and $0 \leq \lambda \leq 1$, then

$$\begin{aligned}(f_1 + f_2)(\lambda y + (1-\lambda)z) &= f_1(\lambda y + (1-\lambda)z) + f_2(\lambda y + (1-\lambda)z)\\ &\leq \lambda f_1(y) + (1-\lambda)f_1(z) + \lambda f_2(y) + (1-\lambda)f_2(z)\\ &= \lambda(f_1 + f_2)(y) + (1-\lambda)(f_1 + f_2)(z).\end{aligned}$$

Hence, $(f_1 + f_2)$ is convex on C. Moreover, it is clear from this computation that if either f_1 or f_2 is strictly convex, then $(f_1 + f_2)$ is strictly convex because strict convexity of either function introduces a strict inequality at the right place.

(b) This result follows by an argument similar to that used in (a).

(c) If y, z belong to C and $0 \leq \lambda \leq 1$, then

$$f(\lambda y + (1-\lambda)z) \leq \lambda f(y) + (1-\lambda)f(z)$$

since f is convex on C. Consequently, since ϕ is an increasing, convex function on the range of f, it follows that

$$\begin{aligned}\phi(f(\lambda y + (1-\lambda)z)) &\leq \phi(\lambda f(y) + (1-\lambda)f(z))\\ &\leq \lambda\phi(f(y)) + (1-\lambda)\phi(f(z)).\end{aligned}$$

Thus, the composite function $\phi \circ f$ is convex on C. If f is strictly convex and ϕ is strictly increasing, the first inequality in the preceding computation is strict for $y = z$ and $0 < \lambda < 1$, so $\phi \circ f$ is strictly convex on C.

■

Examples

(a) The function f defined on $\mathbb{R}^3$ by

$$f(x_1, x_2, x_3) = e^{x_1^2 + x_2^2 + x_3^2}$$

is strictly convex.

At first glance, it might semm that the most direct path to verify that f is strictly convex on $\mathbb{R}^3$ would be to show that the Hessian $Hf(x)$ of f is positive definite on $\mathbb{R}^3$. However, the Hessian turns out to be

$$Hf(x) = \begin{pmatrix} (2+4x_1^2)e^{x_1^2+x_2^2+x_3^2} & 4x_1x_2e^{x_1^2+x_2^2+x_3^2} & 4x_1x_3e^{x_1^2+x_2^2+x_3^2} \\ 4x_1x_2e^{x_1^2+x_2^2+x_3^2} & (2+4x_2^2)e^{x_1^2+x_2^2+x_3^2} & 4x_2x_3e^{x_1^2+x_2^2+x_3^2} \\ 4x_1x_3e^{x_1^2+x_2^2+x_3^2} & 4x_2x_3e^{x_1^2+x_2^2+x_3^2} & (2+4x_3^2)e^{x_1^2+x_2^2+x_3^2} \end{pmatrix}$$

Obviously, proving that the Hessian is positive definite for all $x \in \mathbb{R}^n$ will involve quite tedious algebra. No matter there is much simpler way to handle the problem.

First note that

$$h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

is strictly convex since its Hessian

$$Hh(x_1, x_2, x_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is obviously positive definite. Also, $\phi(t) = e^t$ is strictly increasing (since $\phi'(t) = e^t > 0$ for all $t \in \mathbb{R}$) and strictly convex (since $\phi''(t) = e^t > 0$ for all $t \in \mathbb{R}$). Therefore by Theorem 3.4.6(c), $f = \phi \circ h$ is strictly convex on $\mathbb{R}^3$.

(b) Suppose $a^{(1)}, \ldots, a^{(m)}$ are fixed vectors in $\mathbb{R}^n$ and that $c_1, \ldots, c_m$ are positive real numbers. Then the function f defined on $\mathbb{R}^n$ by

$$f(x) = \sum_{i=1}^{m} c_i e^{a^{(i)}\cdot x}$$

is convex.

To prove this statement, first observe that the functions g_i on $\mathbb{R}^n$ defined by

$$g_i(x) = a^{(i)} \cdot x, \quad i = 1, \ldots, m$$

are linear and therefore convex on $\mathbb{R}^n$. Since $h(t) = e^t$ is increasing and convex on $\mathbb{R}$, it follows from theorem 3.4.6(c) that the functions

$$h(g_i(x)) = e^{a^{(i)}\cdot x}, \quad i = 1, \ldots, m$$

are all convex on $\mathbb{R}^n$. Since $c_1, \ldots, c_m$ are positive real numbers, you can apply Theorem 3.4.6(a) and (b) to conclude that

$$f(x) = \sum_{i=1}^{m} c_i e^{a^{(i)}\cdot x}$$

is convex on $\mathbb{R}^n$.

(c) The function f defined on $\mathbb{R}^2$ by

$$f(x_1, x_2) = x_1^2 - 4x_1x_2 + 5x_2^2 - \ln x_1x_2$$

is strictly convex on $C = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$.

In fact $f(x) = g(x) + h(x)$ where

$$g(x_1, x_2) = x_1^2 - 4x_1x_2 + 5x_2^2, \quad h(x_1, x_2) = -\ln(x_1x_2)$$

so Theorem 3.4.6(a) will imply that f is strictly convex once you are able to show that g and h are convex and at least one of these functions is strictly convex on C. But the Hessian of g is

$$\begin{pmatrix} 2 & -4 \\ -4 & 10 \end{pmatrix}$$

principal minors of this matrix are $\Delta_1 = 2$, $\Delta_2 = 4$, g is strictly convex on $\mathbb{R}^2$. Consequently, all that you need to do now is to show that h is convex on C. But

$$h(x_1, x_2) = -\ln x_1 - \ln x_2$$

and the function $\phi(t) = -\ln t$ $(t > 0)$ is strictly convex since $\phi''(t) = 1/t^2$, so h is convex on C by Theorem 3.4.6(c).

3.5 Conclusion

In this section, you looked at *Unconstrained* optimization problem. You learnt the *first order necessary optimality condition* and the *second order necessary and sufficient optimality condition*. You were also introduced to the notion of convex sets and convex functions. And you proved some results in optimization problems defined on a convex set.

3.6 Summary

Having gone through this unit, you now know the following

(i) $\bar{x}$ is a *local minimizer* of f in D if there exists $r > 0$ such that

$$f(\bar{x}) \leq f(x) \quad \text{for all } x \in D \cap B(\bar{x}, r)$$

(ii) $\bar{x}$ is a *global minimizer* of f in D if

$$f(\bar{x}) \leq f(x) \quad \text{for all } x \in D$$

Reversing the inequalities in (i) and (ii) gives you the the definitions of *local maximizer* and *global maximizer* respectively of f. You also have the definition of strict optimas' if the inequalities are made to be strict.

(iii) If $\overline{x}$ is a local minimizer, then $\overline{x}$ is a critical point, (i.e., $\nabla \boldsymbol{f}(\overline{x}) = 0$). This is the *first order necessary* optimality condition.

(iv) $\overline{x}$ is a local minimizer if and only if the hessian of $\boldsymbol{f}$ at $\overline{x}$ i.e., $H\boldsymbol{f}(\overline{x})$ is semipositive definite. This the *second order necessary and sufficient* optimality condition.

(v) If $\boldsymbol{f}$ is a convex function, then every local minimizer is also a global minimizer. In addition if $\boldsymbol{f}$ is a strictly convex function, then $\overline{x}$ is a unique global minimizer

3.7 Tutor Marked Assignments (TMAs)

Exercise 3.7.1

1. Find the local and global minimizers and maximizers of the following functions

   (a) $\boldsymbol{f}(x) = x^2 + 2x.$

   (b) $\boldsymbol{f}(x) = x^2 e^{-x^2}.$

   (c) $\boldsymbol{f}(x) = x^4 + 4x^3 + 6x^2 + 4x.$

   (d) $\boldsymbol{f}(x) = x + \sin x.$

2. Classify the following matrices according to whether they are positive or negative definite or semidefinite or indefinite.

   (a) $\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{array}$

   (b) $\begin{array}{ccc} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{array}$

   (c) $\begin{array}{ccc} 7 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 5 \end{array}$

   (d) $\begin{array}{ccc} 3 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{array}$

   (e) $\begin{array}{ccc} -4 & 0 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & -5 \end{array}$

   (f) $\begin{array}{ccc} 2 & -4 & 0 \\ -4 & 8 & 0 \\ 0 & 0 & -3 \end{array}$

3. Write the quadratic form $Q_A(x)$ associated with each of the following matrices A :

(a) $A = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & -3 \\ -3 & 0 \end{pmatrix}$

(c) $\begin{matrix} 1 & -1 & 0 \\ -1 & -2 & 0 \\ 0 & 2 & 3 \end{matrix}$

(d) $\begin{matrix} -3 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 4 \end{matrix}$

4. Write the following quadratic forms in the form $x^T Ax$ where A is an appropriate symmetric matrix.

   (a) $3x_1^2 - x_1x_2 + 2x_2^2$.

   (b) $x_1^2 + 2x_2^2 - 3x_3^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3$.

   (c) $2x_1^2 - 4x_3^2 + x_1x_2 - x_2x_3$.

5. Suppose f is defined on $\mathbb{R}^3$ by

$$f(x) = c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_4x_1x_2 + c_5x_1x_3 + c_6x_2x_3.$$

   Show that f is the quadratic form associated with $\frac{1}{2}Hf$. Discuss generalizations to higher dimensions.

6. Show that the principal minors of the matrix

$$A = \begin{pmatrix} 1 & -8 \\ 1 & 1 \end{pmatrix}$$

   are positive, but that there are $x = 0$ in $\mathbb{R}^2$ such that $x^T Ax < 0$. What conclusion can you draw from this?

7. Use the principal minor criteria to determine (if possible) the nature of the critical points of the following functions:

   (a) $f(x_1, x_2) = x_1^3 + x_2^3 - 3x_1 - 12x_2 + 20$.

   (b) $f(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3$.

   (c) $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 4x_1x_2$.

   (d) $f(x_1, x_2) = x_1^4 + x_2^4 - x_1^2 - x_2^2 + 1$.

   (e) $f(x_1, x_2) = 12x_1^3 - 36x_1x_2 - 2x_2^3 + 9x_2^2 - 72x_1 + 60x_2 + 5$.

8. Show that the functions

$$f(x_1, x_2) = x_1^2 + x_2^3,$$

   and

$$g(x_1, x_2) = x_1^2 + x_2^4.$$

   both have a critical point at (0, 0), both have positive semidefinite Hessians at (0, 0), but (0, 0) is a local minimizer for $g(x_1, x_2)$ but not for $f(x_1, x_2)$.

9. Find the global maximizers and minimizers, if they exist, for the following functions:
   (a) $f(x_1, x_2) = x_1^2 - 4x_1 + 2x_2^2 + 7$.
   (b) $f(x_1, x_2) = e^{-(x_1^2+x_2^2)}$.
   (c) $f(x_1, x_2) = x_1^2 - 2x_1x_2 + \frac{1}{3}x_2^3 - 4x_2$.
   (d) $f(x_1, x_2, x_3) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$.
   (e) $f(x_1, x_2) = x_1^4 + 16x_1x_2 + x_2^8$.
10. Show that although $(0, 0)$ is a critical point of $f(x_1, x_2) = x_1^5 - x_1x_2^6$, it is neither a local maximizer nor a local minimizer of $f(x_1, x_2)$.
11. Define $f(x, y)$ on $\mathbb{R}^2$ by
$$f(x, y) = x^4 + y^4 - 32y^2$$
   (a) Find a point in $\mathbb{R}^2$ at which Hf is indefinite.
   (b) Show that $f(x, y)$ is coercive.
   (c) Minimize $f(x, y)$ on $\mathbb{R}^2$.
12. Define $f(x, y, z)$ on $\mathbb{R}^3$ by
$$f(x, y, z) = e^x + e^y + e^z + 2e^{-x-y-z}$$
   (a) Show that $Hf(x, y, z)$ is positive definite at all points of $\mathbb{R}^3$.
   (b) Show that $(\ln 2/4, \ln 2/4, \ln 2/4)$ is the strict global minimizer of $f(x, y, z)$ on $\mathbb{R}^3$.
13. (a) Show that no matter what values of a is chosen, the function
$$f(x_1, x_2) = x_1^3 - 3ax_1x_2 + x_2^3$$
   has no global maximizers.
   (b) Determine the nature of the critical points of this function for all values of a.

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UNIT 4

CONSTRAINED OPTIMIZATION

4.1 Introduction

It is not often that optimization problems have unconstrained solutions. Typically, some or all of the constraints will matter. Through out this unit, you will be concerned with examining necessary conditions for optima in such a context.

4.2 Objectives

At the end of this unit, you should be able to

(i) Give the definition of a constrained optimization problem.

(ii) Solve Equality constained problems.

(iii) Apply the Lagrange's theorem.

(iv) State and apply the first order necessary conditions.

(v) State and apply the second order necessary and sufficient conditions.

(vi) Solve Inequality constrained problems

4.3 Constrained Optimization Problem

Just as defined in unit 10, An optimization problem is called ***constrained*** if it is of the form

$$\begin{array}{ll} \min(\text{or } \max) & f(x) \\ \text{Subject to:} & h_i(x) = 0,\ i = 1, \ldots, l \\ & g_i(x) \geq 0,\ i = 1, \ldots, m \\ & x \in U. \end{array} \tag{4.1}$$

Where $f : U \to \mathbb{R}$, U is an open set of $\mathbb{R}^n$ is called the *Objective function*, $g_1, \ldots, g_k, h_1, \ldots, h_l : U \subset \mathbb{R}^n \to$ are the *constraint functions.*

If you define $g = (g_1, \ldots, g_k) : \mathbb{R}^n \to \mathbb{R}^k$ and $h = (h_1, \ldots, h_l) : \mathbb{R}^n \to \mathbb{R}^l$, then you can rewrite the constrained problem as follows

$$\begin{array}{ll} \min(\text{or } \max) & f(x) \\ \text{Subject to:} & h(x) = 0 \\ & g(x) \geq 0 \\ & x \in U. \end{array} \tag{4.2}$$

If you define in the sequel that the constraint set D as

$$D = U \cap \{x \in \mathbb{R}^n : h(x) = 0,\ g(x) \geq 0\}, \tag{4.3}$$

Then, Problem (4.2) reduces to

$$\begin{array}{ll} \min(\text{or}) \max) & f(x) \\ \text{Subject to:} & x \in D \end{array} \tag{4.4}$$

Many problems in economic theory can be written in this form. For example you can readily see that if f, g and h are linear functions, then the problem (4.2) becomes a linear programming problem, to which, if solution exist, you can use the simplex method, discussed in previous units, to solve. Nonnegativity constraints are easily handled: if a problem requires that $x \in \mathbb{R}^n_+$, this may be accomplished by defining the function $h_j : \mathbb{R}^n \to \mathbb{R}$

$$g_j(x) = x_j, \qquad j = 1, \ldots, n,$$

and using the n inequality constraints

$$g_j(x) \geq 0$$

More generally, requirements of the form $\alpha(x) \geq a$, $\beta(x) \leq b$, or $\psi(x) = c$ (where a, b and c are constants), can all be expressed in the desired form by simply writing them as $\alpha(x) - a \geq 0$, $b - \beta(x) \geq 0$, or $c - \psi(x) = 0$.

Your study in this unit, is divided into two parts namely;

1. Equality-Constrained optimization problems.

2. Inequality-constrained optimization problems.

You will now take it one after the other and study them.

4.4 Equality-Constraint

Coming back to the study of minimization with constraints. More specifically, you will tackle, in this section, the following problem

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{subject to} \quad & h_1(x) = 0 \\ & h_2(x) = 0 \\ & \vdots \\ & h_m(x) = 0 \end{aligned} \tag{4.5}$$

where $x \in D \subset \mathbb{R}$, and the function $f, h_1, h_2, \ldots, h_m$ are continuous, and usually assumed to be in C^2 (i.e., with continuous second partial derivatives).

Observe that when f and h_j's are linear, the problem is a linear programming one and can be solved using the simplex algorithm. Hence you would like to focus on the case that these functions are nonlinear.

In order to gain some intuition, you can consider the case where $n = 2$ and $m = 1$. The problem becomes

$$\begin{aligned} \text{minimize} \quad & f(x, y) \\ \text{subject to} \quad & h(x, y) = 0, \quad (x, y) \in \mathbb{R}^2. \end{aligned}$$

The constraint $h(x, y) = 0$ defines a curve as shown below. Differentiate the equation with respect to x :

$$\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\frac{dh}{dx} = 0.$$

The tangent of the curve is $T(x, y) = (1, \frac{dy}{dx})$. And the gradient of the curve is $\nabla h = (\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y})$. So the above equation states that

$$T \cdot \nabla h = 0;$$

namely, the tangent of the curve must be normal to the gradient at all the time. Suppose you are at a point on the curve. To stay on the curve, any motion must be along the tangent T.

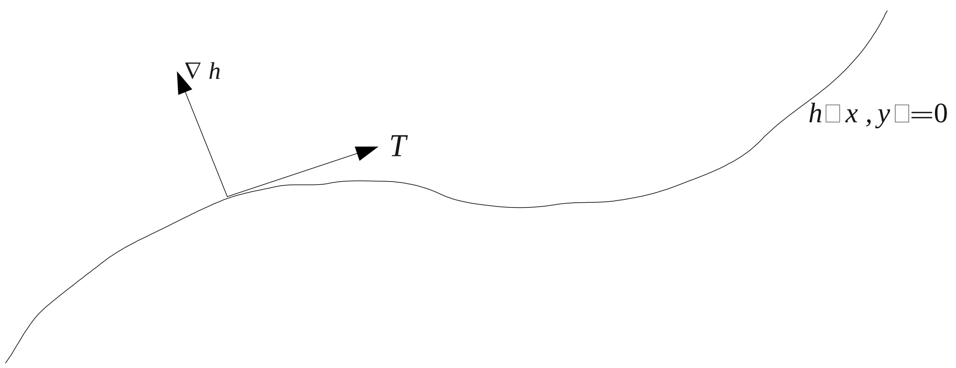

Figure 4.1:

In order to increase or decrease $\boldsymbol{f}(x, y)$, motion along the constraint curve must have a component along the gradient of f, that is,

$$\nabla \boldsymbol{f} \cdot T = 0.$$

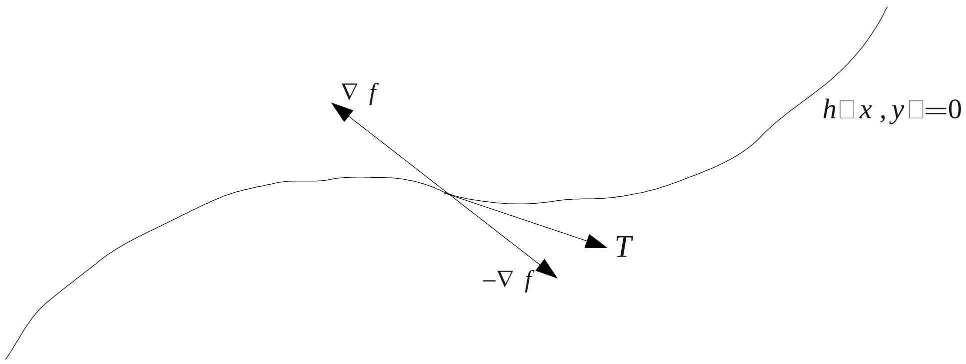

Figure 4.2:

At an extremum of $\boldsymbol{f}$, a differential motion should not yield a component of motion along $\nabla \boldsymbol{f}$. Thus T is orthogonal to $\nabla \boldsymbol{f}$; in other words, the condition

$$\nabla \boldsymbol{f} \cdot T = 0$$

must hold. Now T is orthogonal to bot gradients $\nabla \boldsymbol{f}$ and ∇h at an extrema. This means that $\nabla \boldsymbol{f}$ and ∇h must be parallel. Phrased differently, there exists some $\lambda \in$ R such that

$$\nabla \boldsymbol{f} + \lambda \nabla h = 0. \tag{4.6}$$

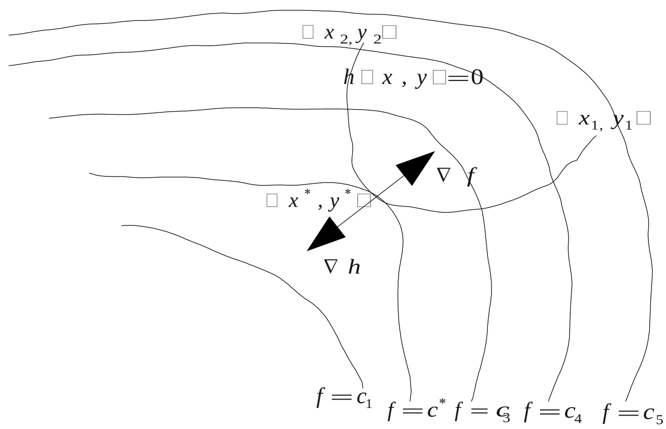

Figure 4.3:

the figure above explains condition (4.6) by superposing the curve $h(x, y) = 0$ onto the family of *level curves* of $\boldsymbol{f}(x, y)$, that is, the collection of curves $\boldsymbol{f}(x, y) = c$, where c is any real number in the range of $\boldsymbol{f}$. In the figure, $c_5 > c_4 > c_3 > c^* > c_1$. The tangent of a level curve is always orthogonal to the gradient ∇f. Otherwise moving along the curve would result in an increase or decrease of the value of f. Imagine a point moving on the curve $h(x, y) = 0$ from (x_1, y_1) to (x_2, y_2). Initially, the motion has a component along the negative gradient direction $-\nabla f$, resulting in the decrease of the value of f. This component becomes smaller and smaller. When the moving point reaches (x^*, y^*), the motion is orthogonal to the gradient. From that point on, the motion starts having a component along the gradient ∇f so the value of f increases. Thus at (x^*, y^*), f achieves its local minimum. The motion is in the tangential direction of the curve $h(x, y) = 0$, which is orthogonal to the gradient ∇h. Therefore at the point (x^*, y^*) the two gradients ∇f and ∇h must be collinear. This is what equation (4.6) says. Let c^* be the local minimum achieved at (x^*, y^*). It is clear that the two curves $f(x, y) = c^*$ and $h(x, y) = 0$ are tangent at (x^*, y^*).

Suppose you find the set S of points satisfying the equations

$$h(x, y) = 0$$
$$\nabla f + \lambda \nabla h = 0 \quad \text{for some } \lambda$$

Then S contains the external points of f to the constraints $h(x, y) = 0$. The above two equations constitute a nonlinear system in the variables x, y, λ. It can be solved using numerical techniques, for example, Newton's method.

4.4.1 Lagrangian

It is convenient to introduce the *Lagrangian* associated with the constrained problem, defined as

$$F(x, y, \lambda) = f(x, y) + \lambda h(x, y)$$

Note

$$\nabla F = \begin{pmatrix} \frac{\partial f}{\partial x} + \lambda \frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial h}{\partial y} \\ h \end{pmatrix}^T = (\nabla f + \lambda h, h).$$

Thus setting $\nabla F = 0$ yields the same system of nonlinear equations you derived earlier.

The value λ is known as the *Lagrange multiplier*. The approach of constructing the Lagrangians and setting its gradient to zero is know as the method of Lagrange multipliers.

Example 4.4.1 Find the extremal values of $f(x, y) = xy$ subject to the constraint

$$h(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

☞ ***Solution.*** First construct the Lagrangian and find its gradient:

$$F(x, y, \lambda) = xy + \lambda \left(\frac{x^2}{8} + \frac{y^2}{2} - 1 \right),$$

$$\nabla F(x, y, \lambda) = \begin{pmatrix} y + \frac{\lambda x}{4} \\ x + \lambda y \\ \frac{x^2}{8} + \frac{y^2}{2} - 1 \end{pmatrix} = 0$$

The above leads to three equations

$$y + \frac{\lambda x}{4} = 0, \tag{4.7}$$
$$x + \lambda y = 0, \tag{4.8}$$
$$x^2 + 4y^2 = 8. \tag{4.9}$$

combining (4.7) and (4.8) yields

$$\lambda^2 = 4 \quad \text{and} \quad \lambda = \pm 2$$

Thus $x = \pm 2y$. Substituting this equation into (4.9) gives you

$$y = \pm 1 \quad \text{and} \quad x = \pm 2.$$

So there are four extremal points of $\boldsymbol{f}$ subject to the constraint h : $(2, 1)$, $(-2, -1)$, $(2, -1)$, and $(-2, -1)$. The maximum value 2 is achieved at the first two points while the minimum value -2 is achieved at the last two points.

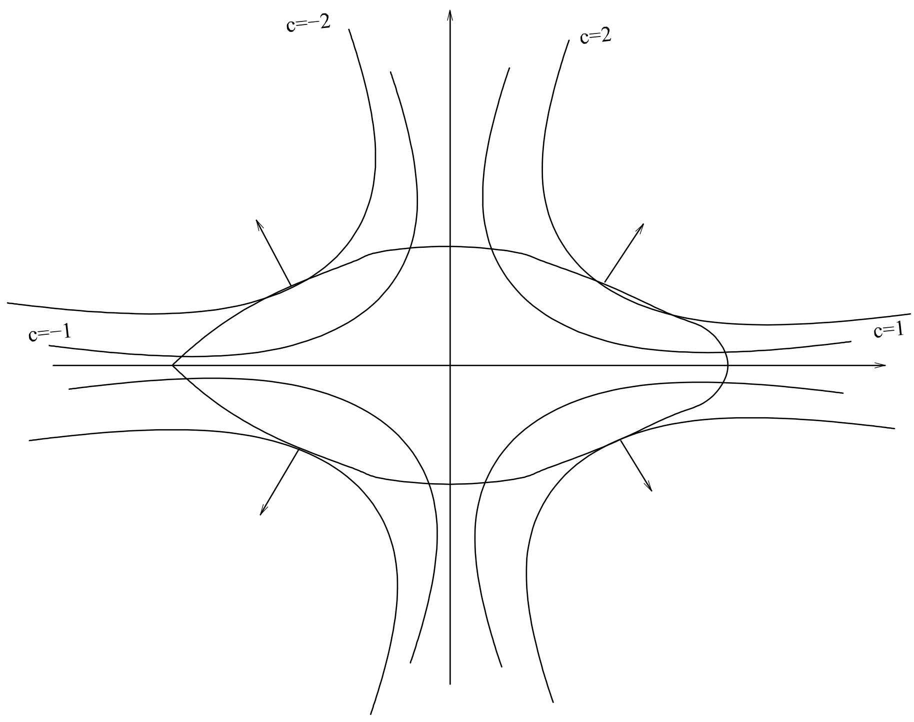

Figure 4.4:

Graphically, the constraint h defines an ellipse. The constraint contours of $\boldsymbol{f}$ are the hyperbolas $xy = c$, with $|c|$ increasing as the curves move out from the origin. ✍

4.4.2 General Formulation

Now you would generalize to the case with multiple constraints. Let $h = (h_1, \ldots, h_m)^T$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Consider the constrained optimization problem below.

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{f}(x) \\ &\text{subject to} \quad h(x) = 0 \end{aligned}$$

Each constraint equation $h_i(x) = 0$ defines a constraint *hypersurface* S in the space $\mathbb{R}^n$. And this surface is smooth provided $h_j(x) \in C^1$.

A curve on S is a family of points $x(t) \in S$ with $a \le t \le b$. The curve is differentiable if $\frac{dx(t)}{dt}$ exists, and twice differentiable if $\frac{d^2x}{dt^2}$ exists. The curve passes through a point x^* if $x^* = x^*(t)$ for some t^*, $a \le t^* \le b$.

The *tangent space* at x^* is the subspace of $\mathbb{R}^n$ spanned by the tangents $\frac{dx}{dt}(t^*)$ of all curves $x(t)$ on S such that $x(t^*) = x^*$. In other words, the tangent space is the set of the derivatives at x^* of all surface curves through x^*. Denote this subspace as T.

A point x satisfying $h(x) = 0$ is a *regular point* of the constraint if the gradient vectors $\nabla h_1(x), \ldots, \nabla h_m(x)$ are linearly independent.

From your previous intuition, you would expect that $\nabla \boldsymbol{f} \cdot v = 0$ for all $v \in T$ at an extremum. This implies that $\nabla \boldsymbol{f}$ lies in the orthogonal complement $T^{\perp}$ of T. Claim that $\nabla \boldsymbol{f}$ can be composed from a linear combination of the ∇h_i's. This is only valid provided that these gradients span $T^{\perp}$, which is true when the extremal point is regular.

Theorem 4.4.1 *At a regular point x of the surface S defined by $h(x) = 0$, the tangent space is the same as*

$$\{y = |\nabla h(x)y = 0\}$$

where the matrix

$$\nabla h = \begin{bmatrix} \nabla h_1 \\ \vdots \\ \nabla h_m \end{bmatrix}.$$

The rows of the matrix $\nabla h(x)$ are the gradient vectors $\nabla h_j(x)$, $j = 1, \ldots, m$. The theorem says that the tangent space at x is equal to the nullspace of $\nabla h(x)$. Thus its orthogonal complement $T^{\perp}$ must equal the row space of $\nabla h(x)$. Hence the vectors $\nabla h_j(x)$ span $T^{\perp}$.

Example 4.4.2 Suppose $h(x_1, x_2) = x_1$. Then $h(x) = 0$ yields the x_2 axis. And $\nabla h = (1, 0)$ at all points. So every $x \in \mathbb{R}^2$ is regular. The tangent space is also the x_2 axis and has dimension 1. If instead $h(x_1, x_2) = x_1^2$, then $h(x) = 0$ still defines the x_2 axis. On this $\nabla h = (2x_1, 0) = (0, 0)$. Thus no point is regular. The dimension of T, which is the x_2 axis, is still one, but the dimension of the space $\{y|\nabla h \cdot y = 0\}$ is two.

Lemma 4.4.1 *Let x^* be a local extremum of $\boldsymbol{f}$ subject to the constraints $h(x) = 0$. Then for all y in the tangent space of the constraint surface at x^*,*

$$\nabla \boldsymbol{f}(x^*)y = 0.$$

The next theorem states that the Lagrange multiplier method as a necessary condition on an extremum point.

Theorem 4.4.2 (First-Order Necessary Conditions) *Let x^* be a local extremum point of $\boldsymbol{f}$ subject to the constraints $h(x) = 0$. Assume further that x^* is a regular point of these constraints. Then there is a $\lambda \in \mathbb{R}^n$ such that*

$$\nabla \boldsymbol{f}(x^*) + \lambda^T \nabla h(x^*) = 0.$$

The first order necessary conditions together with the constraints

$$h(x^*) = 0$$

give a total of $n + m$ equations in $n + m$ variables x^* and λ. Thus a unique solution can be determined at least locally.

Example 4.4.3 You can construct a cardboard box of maximum volume, given a fixed area of cardboard.
Denoting the dimension of the box by x, y, z, the problem can be expressed a

$$\begin{aligned} &\text{maximize} \quad xyz \\ &\text{subject to} \quad xy + yz + xz = \tfrac{c}{2}, \end{aligned}$$

where $c > 0$ is the given area of cardboard. Consider the Lagrangian $xyz + \lambda(xy + yz + xz - \frac{c}{2})$. The first-order necessary conditions are easily found to be

$$yz + \lambda(y + z) = 0, \tag{4.10}$$

$$xz + \lambda(x + z) = 0, \tag{4.11}$$

$$xy + \lambda(x + y) = 0. \tag{4.12}$$

together with the original constraint. Before solving the equation above, note that their sum is

$$(xy + yz + xz) + 2\lambda(x + y + z) = 0,$$

which, given the constraint, becomes

$$c/2 + 2\lambda(x + y + z) = 0.$$

Hence it is clear that $\lambda = 0$. Neither of x, y, z can be zero since if either is zero, all must be so according to (4.10)-(4.12).
To solve the equations (4.10)-(4.12), multiply (4.10) by x and (4.11) by y, and then subtract the two to obtain

$$\lambda(x - y)z = 0$$

Operate similarly on the second and third to obtain

$$\lambda(y - z)x = 0.$$

Since no variables can be zero, it follows that

$$x = y = z = \sqrt{\frac{c}{6}}$$

is the unique solution to the necessary conditions. The box must be a cube.

You can derive the second-order conditions for constrained problems, assuming f and h are twice continuously differentiable.

Theorem 4.4.3 (Second-Order Necessary Conditions) *Suppose that x^* is a local minimum of f subject to $h(x) = 0$ and that x^* is a regular point of these constraints. Then there is a $\lambda \in R^m$ such that*

$$\nabla f(x^*) + \lambda^T \nabla h(x^*) = 0.$$

The matrix

$$L(x^*) = Hf(x^*) + \sum_{i=1}^{m} \lambda_i H h_i(x) \tag{4.13}$$

is positive semidefinite on the tangent space $\{y | \nabla h(x^*) y = 0\}$.

Theorem 4.4.4 (Second-Order Sufficient Conditions) *Suppose there is a point x^* satisfying $h(x^*) = 0$, and a λ such that*

$$\nabla f(x^*) + \lambda^T h(x^*) = 0.$$

Suppose also that the matrix $L(x^*)$ defined in (4.13) is positive definite on the tangent space $\{y | \nabla h(x^*) y = 0\}$. Then x^* is a strict local minimum of f subject to $h(x) = 0$.

Example 4.4.4 Consider the problem

$$\begin{array}{ll} \text{minimize} & x_1x_2 + x_2x_3 + x_1x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 3 \end{array}$$

The first order necessary conditions become

$$\begin{aligned} x_2 + x_3 + \lambda &= 0 \\ x_1 + x_3 + \lambda &= 0 \\ x_1 + x_2 + \lambda &= 0. \end{aligned}$$

You can solve these equations together with the one constraint equation and obtain

$$x_1 = x_2 = x_3 = 1 \quad \text{and} \quad \lambda = -2$$

Thus $x^* = (1, 1, 1)^T$.

Now you need to resort to the second-order sufficient conditions to determine if the problem achieves a local maximum and minimum at $x_1 = x_2 = x_3 = 1$. You will find the matrix

$$\begin{aligned} L(x^*) &= Hf(x^*) + \lambda H h(x^*) \\ &= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{aligned}$$

is neither positive nor negative definite. On the tangent space $M = \{y | y_1 + y_2 + y_3 = 0\}$, however, you note that

$$\begin{aligned} y^T L y &= y_1(y_2 + y_3) + y_2(y_1 + y_3) + y_3(y_1 + y_2) \\ &= -(y_1^2 + y_2^2 + y_3^2) \\ &< 0, \qquad \text{for all } y = 0. \end{aligned}$$

Thus L is negative definite on M and the solution 3 you found is atleas a local maximum.

4.5 Inequality Constraints

Finally, you will address the problems of the general form

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{subject to} \quad & h(x) = 0 \\ & g(x) \geq 0 \end{aligned}$$

where $h = (h_1, \ldots, h_m)^T$ and $g = (g_1, \ldots, g_p)^T$.

A fundamental concept that provides a great deal of insight as well as simplifies the required theoretical development is that of an *active constraint*. An inequality constraint $g_i(x) \leq 0$ is said to be *active* at a feasible point x if $g_i(x) = 0$ and *inactive* at x if $g_i(x) = 0$. By convention you refer to any equality constraint $h_i(x) = 0$ as active at any feasible point. The constraints active at a feasible point x restrict the domain of feasibility in neighbourhood of x. Therefore, in studying the properties of a local minimum point, it is clear that attention can be restricted to the active constraints. This is illustrated in the figure below where local properties satisfied by the solution x^* obviously do not depend on the inactive constraints g_2 and g_3.

Assume that the function $f, h = (h_1, \ldots, h_m)^T, g = (g_1, \ldots, g_p)^T$ are twice continuously differentiable. Let x^* be a point satisfying the constraint.

$$h(x^*) = 0 \text{ and } g(x^*) \leq 0,$$

and let $J = \{j | g_j(x^*) = 0\}$. Then x^* is said to be a *regular point* of the above constraints if the gradient vectors $\nabla h_i(x^*)$, $\nabla g_j(x^*)$, $1 \leq i \leq m$, $j \in J$ are linearly independent. Now suppose this regular point x^* is also a relative minimum point for the original problem (4.6). Then it is shown that there exists a vector $\lambda \in \mathbb{R}^m$ and a vector $\mu \in \mathbb{R}^p$ with $\mu \geq 0$ such that

$$\nabla f(x^*) + \lambda^T \nabla h(x^*) + \mu^T \nabla g(x^*) = 0$$

$$\mu^T g(x^*) = 0$$

Since $\mu \geq 0$ and $g(x^*) \leq 0$, the second constraint above is equivalent to the statement that a component of μ may be nonzero only if the corresponding constraint is active. To find a solu- tion, you can enumerate various combinations of *active* constraints, that is, constraints where equalities are attained at x^*, and check the signs of the resulting Lagrangian multipliers.

There are a number of distinct theories concerning this problem, based on various regular- ity conditions or constraint qualifications, which are directed toward obtaining definite general statements of necessary and sufficient conditions. One can by no means pretend that all such re- sults can be obtained as minor extensions of the theory for problems having equality constraints only. To date, however, their use has been limited to small-scale programming problems of two or three variables.

4.6 Conclusion

In this unit, you were introduced to constrained optimization problems, which could be equality, inequality, or mixed constraints. You looked at the theorem of Lagrange for local optimum of a constrained problem.

4.7 Summary

Having gone through this unit, you now

(i) define equality and inequality constrained optimization problem.

(ii) state and use the lagrange theorem.

(iii) State and apply the First-Order Necessary Conditions.

(iv) State and apply the second-order necessary and sufficient conditions.

4.8 Tutor Marked Assignments(TMAs)

Exercise 4.8.1

1. Find the minimum and maximum of $f(x, y) = x^2 - y^2$ on the unit circle $x^2 + y^2 = 1$ using the Lagrange multipliers method. Using the substitution $y^2 = 1 - x^2$, solve the same problem as a single variable unconstrained proble. Do you get the same results? Why or Why not?
2. Show that the problem of maximizing $f(x, y) = x^3 + y^3$ on the constraint set $D = \{(x, y)|x + y = 1\}$ has no solution. Show also that if the Lagrangian method were used on this problem, the critical points of the Lagrangian have a unique solution. Is the point identified by this solution either a local maximum or a (local or global) minimum?
3. Find the maxima and minima of the following functions subject to the specified constraints:

   (a) $f(x, y) = xy$ subject to $x^2 + y^2 = 2a^2$.

   (b) $f(x, y) = 1/x + 1/y$ subject to $(1/x)^2 + (1/y)^2 = (1/a)^2$.

   (c) $f(x, y, z) = x + y + z$ subject to $(1/x) + (1/y) + (1/z) = 1$.

   (d) $f(x, y, z) = xyz$ subject to $x + y + z = 5$ and $xy + xz + yz = 8$.

   (e) $f(x, y, z) = x + y$ for $xy = 16$

   (f) $f(x, y, z) = x^2 + 2y - z^2$ subject to $2x - y = 0$ and $x + z = 6$.
4. Maximize and minimize $f(x, y) = x + y$ on the lemniscate $(x^2 - y^2)^2 = x^2 + y^2$.
5. Consider the problem

$$\min x^2 + y^2 \text{ subject to } (x - 1)^3 - y^2 = 0.$$

   (a) Solve the problem geometrically.

(b) Show that the method of Lagrange multipliers does not work in this case. Can you explain why?

6. Consider the following problem where the objective function is quadratic and the constraints are linear
$$\max_x c^T x + \frac{1}{2} x^T D x \quad \text{subject to } Ax = b$$
where c is a given n-vector. D is a given $n \times n$ symmetric, negative definite matrix, and A is a given $m \times n$ matrix.
   (a) Set up the Lagrangean and obtain the first-order condtions.
   (b) Solve for the optimal vector x^* as a function of A, b, c and D.

7. Solve the problem
$$\max f(x) = x^T Ax \quad \text{subject to} \quad x \cdot x = 1$$
where A is a given symmetric matrix.

8. Solve the following maximization problem:
$$\begin{array}{ll} \text{Maximize} & \ln x + \ln y \\ \text{Subject to} & x^2 + y^2 = 1 \\ \text{with} & x, y \geq 0. \end{array}$$

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