



**NATIONAL OPEN UNIVERSITY OF NIGERIA**

**COURSE CODE : ENT 321**

**COURSE TITLE:  
QUANTITATIVE METHODS FOR BUSINESS DECISIONS**



**ENT 321**  
**QUANTITATIVE METHODS FOR BUSINESS DECISIONS**

Course Team      Onyemaechi J. Onwe, Ph.D. (Developer/Writer) – NOUN  
Ibrahim Idrisu (Coordinator) – NOUN  
Onyemaechi J. Onwe, Ph.D. (Prog. Leader) – NOUN



## **NATIONAL OPEN UNIVERSITY OF NIGERIA**

National Open University of Nigeria  
Headquarters  
14/16 Ahmadu Bello Way  
Victoria Island  
Lagos

Abuja Office  
5, Dar es Salaam Street  
Off Aminu Kano Crescent  
Wuse II, Abuja  
Nigeria

e-mail: [centralinfo@nou.edu.ng](mailto:centralinfo@nou.edu.ng)  
URL: [www.nou.edu.ng](http://www.nou.edu.ng)

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## **Introduction**

This course, ENT 321: Quantitative Methods for Business Decisions is a two credit unit compulsory course for students studying Entrepreneurial Development and Small Business Management and related programmes in the School of Business and Human Resources Management.

The course has been conveniently arranged for you in eighteen distinct but related units of study activities. In this course guide, you will find out what you need to know about the aims and objectives of the course, components of the course material, arrangement of the study units, assignments, and examinations.

## **The Course Aim**

The course is aimed at acquainting you with what quantitative techniques are all about and letting you understand the practical applications of quantitative techniques in business and economic decision making. To ensure that this aim is achieved, some important background information will be provided and discussed, including:

- definition of quantitative techniques
- uses of quantitative techniques
- tools and applications of quantitative techniques
- the correlation theory
- forecasting and time-series analysis
- index numbers
- inventory control
- decision analysis
- network planning and analysis
- arithmetic and geometric progression
- interest rate and depreciation
- present value and investment appraisals

## **The Course Objectives**

At the end of the course you should be able to:

1. appreciate the uses and importance of quantitative methods in decision making;
2. formulate and solve decision problems in quantitative terms;
3. discuss business forecasts based on past data;
4. compute real monetary values for investment projects;
5. explain profitable inventory decisions;
6. plan network activities for productive business operations.

## Course Material

The course material package is composed of:

- The Course Guide
- The Study Units
- Self-Assessment Exercises
- Tutor-Marked Assignments
- References/Further Readings

## The Study Units

The study units are as listed below:

### Module 1

- Unit 1 Uses, Importance, and Tools of Quantitative Techniques in Decision Making
- Unit 2 Mathematical Tools I: Equation and Inequalities
- Unit 3 Mathematical Tools I: Simultaneous Equations, Linear Functions, and Linear Inequalities
- Unit 4 Mathematical Tools II: Introduction to Matrix Algebra
- Unit 5 Mathematical Tools III: Applied Differential Calculus

### Module 2

- Unit 1 Statistical Tool I: Measures of Averages
- Unit 2 Statistical Tools II: Measures of Variability or Dispersion
- Unit 3 Statistical Tools III: Sets and Set Operations
- Unit 4 Statistical Tools IV: Probability Theory and Applications
- Unit 5 Correlation Theory

### Module 3

- Unit 1 Forecasting and Time-Series Analysis
- Unit 2 Index Numbers
- Unit 3 Inventory Control
- Unit 4 Decision Analysis
- Unit 5 Network Planning and Analysis

### Module 4

- Unit 1 Arithmetic and Geometric Progression
- Unit 2 Interest Rate and Depreciation
- Unit 3 Present Values and Investment Analysis

## **Assignments**

Each unit of the course has a self assessment exercise. You will be expected to attempt them as this will enable you understand the content of the unit.

## **Tutor-Marked Assignment**

The Tutor-Marked Assignments at the end of each unit are designed to test your understanding and application of the concepts learned. It is important that these assignments are submitted to your facilitators for assessments. They make up 30 percent of the total score for the course.

## **Final Examination and Grading**

At the end of the course, you will be expected to participate in the final examinations as scheduled. The final examination constitutes 70 percent of the total score for the course.

## **Summary**

This course, ENT 321: Quantitative Methods for Business Decisions is ideal for today's computerised business environment. It will enable you apply quantitative techniques in such business functions as planning, controlling, forecasting, and evaluation. Having successfully completed the course, you will be equipped with the latest global knowledge on business decisions. Enjoy the course.

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**MODULE 1**

- Unit 1 Uses, Importance, and Tools of Quantitative Techniques in Decision Making
- Unit 2 Mathematical Tools I: Equation and Inequalities
- Unit 3 Mathematical Tools I: Simultaneous Equations, Linear Functions, and Linear Inequalities
- Unit 4 Mathematical Tools II: Introduction to Matrix Algebra
- Unit 5 Mathematical Tools III: Applied Differential Calculus

**UNIT 1 USES, IMPORTANCE, AND TOOLS OF QUANTITATIVE TECHNIQUES IN DECISION MAKING****CONTENTS**

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Definition and Importance of Quantitative Techniques
    - 3.1.1 Definition of Quantitative Techniques
    - 3.1.2 Importance of Quantitative techniques
  - 3.2 Tools of Quantitative Analysis
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

**1.0 INTRODUCTION**

The complexity of the modern business operations, high costs of technology, materials and labour, as well as competitive pressures and the limited time frame in which many important decisions must be made, all contribute to the difficulty of making effective business economic decisions. These call for the need of decision makers to apply such alternative approaches as quantitative techniques.

In recent times, very few business and economic decisions are made without the application of quantitative techniques. For example, a decision on the location of a new manufacturing plant would be primarily based on such economic factors with quantitative measures as construction costs, prevailing wage rates, taxes, energy and pollution control costs, marketing and transportation costs, and related factors. An understanding of the applicability of quantitative methods to economic decisions is, therefore, of fundamental importance to any economics student.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- appreciate both the meaning and importance of quantitative techniques
- state the uses of quantitative techniques
- identify tools of quantitative analysis.

## 3.0 MAIN CONTENT

### 3.1 Definition and Importance of Quantitative Techniques

#### 3.1.1 Definition of Quantitative Techniques

A quantitative technique can be viewed as a scientific approach to decision making with special emphasis on the quantification rather than qualification of decision variables. To buttress this definition, consider the following management decision on pricing of a new product:

The XYZ Company is in the business of manufacturing and distributing electronics equipment: radios, stereos, etc. The company decides to make a new two-way Citizen's Band (CB) radio. The question is **what should be the price of the new CB radio?**

Through market research and comparison with other products, management agrees that the product could be priced between N60 and N100 and still compete effectively in the market place.

We reduce the choice to the prices of:

N60, N70, N85, and N100.

The decision-making process reduces to the selection of one of these four prices. So what is the **best price**, that is, the price that maximises the profit of the company?

Market research predicts that the quantities sold at the prices of N60, N70, N85, and N100 would be 2600, 2200, 1600, and 1000 units respectively. The price, production and cost schedules are given below:

#### Pricing Alternatives

	1	2	3	4
P	N60	N70	N85	N100
Q	2,600	2,200	1,600	1,000
C	N141,000	N127,000	N106,000	N85,000
R	N156,000	N154,000	N136,000	N100,000
Π	N15,000	N27,000	N30,000	N15,000

By definition,

$R = PQ$  (for each price alternative)

$\Pi = R - C$  (for each price alternative).

From the table, we observed that the best price is the price for which the maximum profit can be obtained. This price is the no. 3 alternative, that is, N88/unit, with profit of N30, 000. The second best price alternative is N70/unit, with profit of N27, 000.

The assumptions under which this solution is optimum are:

1. only the prices N60, N70, N85, and N100 are possible;
2. the corresponding sales and cost are known exactly;
3. the criterion is to maximise profit.

### 3.1.2 Importance of Quantitative Techniques

Today's scholar would want to know if quantitative technique, as a course, is necessary in the study of business and entrepreneurship management. The thinking has been that this is a course to be worried about by students in economics and related studies. The truth is that today's business cannot do without some form of quantitative aptitude. Accounting principles are built on the premises of quantitative techniques. So also are production and pricing principles.

Apart from the above, quantitative techniques are essentially valuable in the following areas of management decisions.

1. Planning
2. Forecasting
3. Control
4. Evaluation

### 3.2 Tools of Quantitative Analysis

Quantitative analysis basically constitutes two tools.

- 1) Mathematical tools
- 2) Statistical tools

It is the aim of our discussion to concentrate on the application of these tools.

**Mathematical Tools** are used in developing mathematical models. The general approach to the development of mathematical models of economic decisions can be outlined as follows:

1. **Formulate the Problem:** Determine the variables involved and categorise them into those over which you have no control and those over which you have control. The latter are referred to as decision variables. Specify all relevant constraints, such as production requirements, costs, and the like.
2. **Establish the Criterion for Optimality:** This involves the definition of an appropriate goal. Optimal solutions involve either profit maximisation or cost minimisation.
3. **Develop the Model:** Determine the specific mathematical relationship which exists among the different variables. These are stated in the form of an equation or a set of equations.
4. **Perform the analysis:** Substitute the values of the uncontrollable variables in the mathematical model and determine the values of the decision variables, which lead to an optimal or a minimum solution.

In practice, the development of a mathematical model should be strongly influenced by the objective for which the model is being constructed.

**Statistical Tools** are complementary to mathematical tools. Statistics can be viewed as the application of scientific model in the analysis of numerical data for the purpose of drawing inferences that are useful in making rational economic decisions.

There are basically two types of statistics:

- 1) **Primary Statistics:** Involving the analysis of primary data.
- 2) **Secondary Statistics:** Involving the analysis of secondary data.

Further classification of statistics includes:

- 1) **Descriptive Statistics** - Involving the collection, presentation and characterisation of a set of data in order to properly describe the various features of the given set of data.
- 2) **Inferential Statistics** – Those tools that are used in the estimation of a characteristic of a population or the making of a decision concerning a population based only upon sample results.

#### 4.0 CONCLUSION

Quantitative technique is a course to be worried about by students in economics and related studies. The truth is that today's business cannot

do without some form of quantitative aptitude. Accounting principles are built on the premises of quantitative techniques. So also are production and pricing principles.

Apart from the above, quantitative techniques are essentially valuable in the following areas of management decisions.

1. Planning
2. Forecasting
3. Control
4. Evaluation

## **5.0 SUMMARY**

This unit provided some background information on the study of quantitative methods. In a nutshell, the followings were the major information obtained from the discussions.

1. A quantitative technique can be viewed as a scientific approach to decision making with special emphasis on the quantification rather than qualification of decision variables.
2. Quantitative analysis basically constitutes two tools:
  - Mathematical tools
  - Statistical tools
3. The development of a mathematical model should be strongly influenced by the objective for which the model is being constructed. Statistical tools are complementary to mathematical tools. Statistics in general, can be viewed as the application of scientific model, in the analysis of numerical data, for the purpose of drawing inferences that are useful in making rational economic decisions.

## **6.0 TUTOR-MARKED ASSIGNMENT**

## **7.0 REFERENCES/FURTHER READING**

Haessler, E. F. and Paul, R. S. (1976). *Introductory Mathematical Analysis for Students of Business and Economics*, (2<sup>nd</sup> edition.) Reston Virginia: Reston Publishing Company.

## UNIT 2 MATHEMATICAL TOOLS I: EQUATIONS AND INEQUALITIES

### CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Linear Equations
  - 3.2 Quadratic Equations
    - 3.3.1 Solving By Factorisation
    - 3.3.2 Solving by the Quadratic Formula
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

### 1.0 INTRODUCTION

An *Equation* is a mathematical model expressing the relationship between variables. An *Inequality*, on the other hand, expresses the differences among variables.

Although there are several types of equations and inequalities, depending on the degree of relationship, our major concern here is on Linear and Quadratic Equations, as well as Linear Inequalities.

### 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain what mathematical equations are all about and how to formulate them
- state the business applications of linear and quadratic equations
- recognise the linear equations that are applicable to business decisions
- formulate equations for the solution of simple decision problems.



### 3.0 MAIN CONTENT

#### 3.1 Linear Equations and Quadratic Equations

#### 3.2 Linear Equations

A linear equation in the variable, X, for example, can be written in the form:

$$aX + b = 0,$$

Where **a** and **b** are constants and **a**  $\neq$  0.

Linear equations are classified as first-degree equations. Unknown variables in a linear equation can be solved for, using simple algebraic operations, as you can observe from the following examples.

#### Examples

Solve for the unknowns in the following linear equations:

(i)  $5X - 6 = 3X$

**Solution:**

$$5X - 3X = 6$$

$$(5-3)X = 6$$

$$2X = 6$$

$$\frac{2X}{2} = \frac{6}{2}$$

$$X = 3$$

(ii)  $2(p + 4) = 7p + 2$

**Solution:**

$$2p + 8 = 7p + 2$$

$$2p - 7p = 2 - 8$$

$$(2 - 7)p = -6$$

$$-5p = -6$$

$$\frac{-5p}{-5} = \frac{-6}{-5}$$

$$p = 6/5 = 1.2$$

(iii)  $\frac{7X + 3}{2} - \frac{9X - 8}{4} = 6$

**Solution:**

$$\frac{4(7X + 3)}{2} - \frac{4(9X - 8)}{4} = 4(6)$$

$$2(7X + 3) - 1(9X - 8) = 24$$

$$14X + 6 - 9X + 8 = 24$$

$$14X - 9X = 24 - 6 - 8$$

$$(14-9)X = 10$$

$$5X = 10$$

$$\frac{5X}{5} = \frac{10}{5}$$

$$X = 2$$

**Applications:**

1. XYZ Company produces product A for which cost (including labour and material) is N6/unit. Fixed cost is N80, 000. Each unit is sold for N10. Determine the number of units which must be sold for the company to earn a profit of N60, 000.

**Solution:**

By definition, Profit =  $\Pi$  = Revenue-Cost

That is  $\Pi = R - C$

Let x represent the level of output, so that:

Total Cost = C = Fixed Cost (FC) + Variable Cost (VC),

That is, C = FC + VC

From the problem, FC = N80,000

The variable cost (VC) per unit produced is N6, so that for x units,

VC = 6X

C = FC + VC = 80,000 + 6X

Revenue (R) = (unit price) (quantity sold)

Thus, R = PX

From the problem, P = 10, so that,

R = PX = 10X

The expected profit ( $\Pi$ ) is:

$\Pi = N60, 000$

It follows that:

$$\Pi = R - C$$

$$60,000 = 10x - (80,000 + 6X)$$

Solving for X, we get:

$$10X - (80,000 + 6X) = 60,000$$

$$10X - 80,000 - 6X = 60,000$$

$$10X - 6X = 60,000 + 80,000$$

$$(10 - 6)X = 140,000$$

$$4X = 140,000$$

$$\underline{4X = 140,000}$$

$$\frac{4}{4} = \frac{140,000}{4}$$

$$X = 35,000$$

Therefore, 35,000 units must be sold to earn a profit of N60, 000.

2. A total of N10, 000 was invested in two business ventures, A and B. At the end of the first year, A and B yielded returns on the original investments of 6 percent and 5.75 percent respectively. How was the original amount allocated if the total amount earned was N588.75?

### **Solution**

Let x = amount invested at 6 percent

Then 10,000-x was invested at 5.75 percent

The interest earnings are therefore:

0.06(x) and 0.0575(10,000-x), which must total N588.75.

It follows that:

$$0.06x + 0.0575(10,000-x) = 588.75$$

Solving for x, we get:

$$0.06x + 575 - 0.0575x = 588.75$$

$$0.06x - 0.0575x = 588.75 - 575$$

$$(0.06 - 0.0575)x = 13.75$$

$$0.0025x = 13.75$$

$$\underline{0.0025x = 13.75}$$

$$\frac{0.0025}{0.0025} = \frac{13.75}{0.0025}$$

$$x = 5,500$$

Thus, N5, 500 was invested at 6 percent, while N (10,000-5,500) = N4, 500 was invested at 5.75 percent.

### 3.2 Quadratic Equations

A Quadratic Equation is an equation of the second degree. It is of the form:

$$aX^2 + bX + c = 0,$$

Where  $a$ ,  $b$ , and  $c$  are constant numbers, and  $a \neq 0$

There are two basic **methods of solving quadratic equations**:

- (1) By Factorisation
- (2) By the use of Quadratic formula.

#### 3.2.1 Solving By Factorisation

This involves the determination of the factors that form the given quadratic equation. This will then make the solutions for the unknown easy to come by, as you can see from the following examples.

#### Examples

- (1) Solve for  $x$  in the quadratic equation:  
 $X^2 + X - 12 = 0$

Factorising the left hand side, we get:

$$\begin{array}{ccc} X & - & 3 \\ & \diagdown & / \\ & X & + & 4 \end{array}$$

$(X - 3)$  and  $(X + 4)$  are the factors, so that  
 $(X - 3)(X + 4) = 0$

For this equation to hold, either  $X - 3 = 0$  or  $X + 4 = 0$

It follows that:

If  $X - 3 = 0$   
then,  $X = 3$  and  
if  $X + 4 = 0$   
then,  $X = -4$

The solution set becomes: [3, - 4]

- (2) The length of a shop space is 8 meters more than the width. If the shop space is 48 square meters, what are the dimensions of the space?

**Solution:**

Let X metres be the width of the shop space. From the given information, Length = (X + 8) meters.

$$\begin{aligned} \text{Area} &= \text{Length} \times \text{width} \\ &= (X+ 8)X \end{aligned}$$

The area is given as 48 square metres, so that:

$$\begin{aligned} 48 &= (X+ 8)X \\ \text{or } (X+ 8)X &= 48 \\ X^2 + 8X &= 48 \\ X^2 + 8X - 48 &= 0 \quad (\text{quadratic equation}) \end{aligned}$$

By factorisation, we get:  
 $(X+12)(X- 4) = 0$

For this equation to hold, either

$$\begin{aligned} X + 12 &= 0, \text{ or } X - 4 = 0 \\ \text{If } X + 12 &= 0, X = -12 \\ \text{If } X - 4 &= 0, X = 4 \end{aligned}$$

The width cannot be negative, so that

$$\begin{aligned} \text{Width} &= 4 \text{ metres} \\ \text{Length} &= (4+8) \text{ metres} = 12 \text{ metres.} \end{aligned}$$

Thus, the dimension of the shop space is  
 (4 x 12) metres.

### 3.3.2 Solving by the Quadratic Formula

If  $ax^2 + bX + c = 0$  is a quadratic equation, then, the solution of X is defined by the quadratic formula:

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the values of x are the roots of the quadratic equation.

**Examples:**

(1) Solve for x in:  
 $4x^2 - 17x + 15 = 0$

**Solution:**

Using the quadratic formula,  $a = 4$ ;  $b = -17$ ;  $c = 15$ . By substitution to the formula, we get:

$$\begin{aligned} X &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-17) \pm \sqrt{(-17)^2 - 4(4)(15)}}{2(4)} \\ &= \frac{17 \pm \sqrt{289 - 240}}{8} \\ &= \frac{17 \pm \sqrt{49}}{8} \\ &= \frac{17 \pm 7}{8} \\ X &= \frac{17+7}{8} \quad \text{or} \quad X = \frac{17-7}{8} \\ X &= 3 \quad \text{or} \quad X = 1.25 \end{aligned}$$

(2) Solve for Y in:  
 $2 + 6\sqrt{2y} + 9y^2$

**Solution:**

By the quadratic formula,  $a = 9$ ;  $b = 6\sqrt{2}$ ,  $c = 2$

By substitution,

$$\begin{aligned} Y &= \frac{-6\sqrt{2} \pm \sqrt{(6\sqrt{2})^2 - 4(9)(2)}}{2(9)} \\ &= \frac{-6\sqrt{2} \pm \sqrt{36(2) - 72}}{18} \\ &= \frac{-6\sqrt{2} \pm \sqrt{0}}{18} \\ Y &= \frac{-6\sqrt{2} + 0}{18} \quad \text{or} \quad Y = \frac{-6\sqrt{2} - 0}{18} \end{aligned}$$

$$\begin{aligned} \text{Thus, } Y &= \frac{-\sqrt{2}}{3} = \frac{1.414}{3} \\ &= 0.471 \end{aligned}$$

**Application:**

1. The board of directors of Chizy Co. Ltd. agrees to redeem some of its bonds in 2 years. At that time, N1, 102,500 will be required for the redemption. Let us suppose N1, 000,000 is presently set aside. At what compound annual rate of interest will the N1, 000,000 have to be invested in order that its future value will be sufficient to redeem the bonds?

**Solution:**

Let  $r$  = annual rate of interest

At the end of the first year, the accumulated amount will be:

$$\begin{aligned} & \text{N1, 000,000} + \text{the interest on it} \\ & = 1,000,000 + 1,000,000 (r) \\ & = 1,000,000 (1+r) \end{aligned}$$

Since the interest is compounded, at the end of the second year, the accumulated amount would be:

$$\begin{aligned} & 1,000,000 (1+r) + \text{interest on it:} \\ & 1,000,000 (1+r) + 1,000,000 (1+r)r \end{aligned}$$

According to the problem, the total value at the end of the second year is:

$$1,000,000 (1+r) + 1,000,000 (1+r)r = 1,102,500$$

Solving for  $r$ , we get:

$$1,000,000 (1+r) + 1,000,000(1+r)r = 1,102,500$$

$$1,000,000 [(1+r) + (1+r)r] = 1,102,500$$

$$1,000,000 [1+r+r^2] = 1,202,500$$

$$1,000,000 (r^2 + 2r + 1) = 1,102,500$$

$$1,000,000 (1+r)(1+r) = 1,102,500$$

$$1,000,000 (1+r)^2 = 1,102,500$$

$$(1+r)^2 = \frac{1,102,500}{1,000,000}$$

$$= 1.103$$

$$1+r = \pm\sqrt{1.103}$$

$$r = -1 \pm \sqrt{1.103} = -1 \pm 1.05$$

$$r = -1 + 1.05 \text{ or } r = -1 - 1.05$$

$$r = 0.05 \text{ or } r = -2.05$$

Interest rate cannot be negative, therefore, the desired rate of interest is -.05 or 5 percent.

### SELF ASSESSMENT EXERCISE 1

A company produces product X at a unit cost of N10. If fixed costs are N450, 000, and each unit sells for N25, how many must be sold for the company to make a positive profit of N510,000.

## 4.0 CONCLUSION

You have been acquainted with the concepts of equalities and inequalities, how they can be formulated, solved and applied to practical business decisions. Specifically, two types of equations were discussed, these are linear equations and quadratic equations. You must also have been exposed to the two basic methods of solving quadratic equations: the use of factors; and, the use of quadratic formula.

## 5.0 SUMMARY

1. Linear equations are classified as first-degree equations. Unknown variables in a linear equation can be solved for, using simple algebraic operations.
2. A quadratic equation is an equation of the second degree. It is of the form:

$$aX^2 + bX + c = 0,$$

where  $a$ ,  $b$ , and  $c$  are constant numbers, and  $a \neq 0$

There are two basic **methods of solving quadratic equations**:

- (i) By factorisation
- (ii) By use of the quadratic formula

## 6.0 TUTOR-MARKED ASSIGNMENT

To produce 1 unit of a new product, a company determines that the cost for material is N2.50 per unit and the cost of labour is N4 per unit. The constant overhead cost is N5, 000. If the cost to a wholesaler is N7.50 per unit, determine the least number of units that must be sold by the company to realise a positive profit.

## 7.0 REFERENCES/FURTHER READING

Haessler, E. F. and Paul, R. S. (1976). *Introductory Mathematical Analysis for Students of Business and Economics*, (2<sup>nd</sup> edition.) Reston Virginia: Reston Publishing Company.



## **UNIT 3 MATHEMATICAL TOOLS I (CONTINUED): SIMULTANEOUS EQUATIONS, LINEAR FUNCTIONS, AND LINEAR INEQUALITIES**

### **CONTENTS**

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Simultaneous Equations
  - 3.2 Linear Functions
  - 3.3 Linear Inequalities
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

### **1.0 INTRODUCTION**

This unit is a continuation of our discussions on mathematical tools. This focuses on the other aspects of equations and inequalities that are useful in business and economic decisions.

### **2.0 OBJECTIVES**

At the end of this unit, you should be able to:

- explain what simultaneous equation systems are all about
- describe how simultaneous equation systems are formulated and solved
- analyse business conditions using linear functions and linear inequalities.

### **3.0 MAIN CONTENT**

#### **3.1 Simultaneous Equations**

A simultaneous equation system is a set of equations with two or more unknown. The mostly used method of solving a simultaneous equation system is the so-called “Addition-Subtraction” or “Elimination” method.

Consider the following example:

Solve for X and Y in the system of equations:

$$3X - 4Y = 13 \quad (1)$$

$$3Y + 2X = 3 \quad (2)$$

**Solution:**

Rearranging equation (1) and (2), we get:

$$3X - 4Y = 13 \quad (1)$$

$$2X + 3Y = 3 \quad (2)$$

Multiplying equation (1) by 2 (the coefficient of X in equation (2)) and equation (2) by 3 (the coefficient of X in equation (1)), we get:

$$6X - 8Y = 26 \quad (3)$$

$$6X + 9Y = 9 \quad (4)$$

Subtract equation (4) from equation (3) to get:

$$\begin{array}{r} 6X - 8Y = 26 \\ -(6X + 9Y = 9) \\ \hline 0 - 17Y = 17 \\ -17Y = 17 \\ Y = -1 \end{array}$$

Substituting the value of  $Y = -1$  to equation (1), we get:

$$3X - 4(-1) = 13$$

$$3X + 4 = 13$$

$$3X = 13 - 4$$

$$3X = 9$$

$$X = 3$$

The solution is therefore  $X = 3$ ;  $Y = -1$ .

Note that this substitution will yield the same result if equation (2) is used in the substitution.

**Application:**

1. Suppose a factory manager is setting up a production schedule for two models, A and B, of a new product. Model A requires 4 units of labour input and 9 units of capital input. Model B requires 5 units of labour input and 14 units of capital input. The total available labour input for the production of the product is 335 man-hours per day, and that of capital is 850 units per day. How many of each model should the manager plan to make each day so that all the available labour hours and capital inputs are used.

**Solution:**

Tabulating the schedule, we get:

<b>Input</b>	<b>Model A</b>	<b>Model B</b>	<b>Total Available</b>
Labour	4	5	335
Capital	9	14	850

Let X = number produced of model A per day

Y = number produced of model B per day

These require a total of:

$$4X + 5Y = 335 \quad (\text{labour})$$

and  $9X + 14Y = 850 \quad (\text{capital})$

We now have the simultaneous equations:

$$4X + 5Y = 335 \quad (1)$$

$$9X + 14Y = 850 \quad (2)$$

Solving for X and Y simultaneously, we get:

Multiplying eq. (1) by 9 and eq. (2) by 4:

$$36X + 45Y = 3015 \quad (3)$$

$$36X + 56Y = 3400 \quad (4)$$

eq. (1) – eq. (2):

$$36X + 45Y = 3015$$

$$-(36X + 56Y) = 3400$$

$$0 - 11Y = -385$$

$$-11Y = -385$$

$$Y = \frac{-385}{-11} = 35$$

$$-11$$

Substituting for y in eq. (1):

$$4X + 5(35) = 335$$

$$4X + 175 = 335$$

$$4X = 335 - 175$$

$$4X = 160$$

$$X = 40$$

The solution values,  $Y = 35$  and  $X = 40$  indicate that, according to the specified constraints, the manager should plan to make 40 units of model A and 35 units of model B.

### 3.2 Linear Functions

A function,  $f$ , is said to be a linear function if  $f(x)$  can be written in a form:

$$f(x) = ax + b; a \neq 0$$

The function:  $Y = f(x) = ax + b$  is an equation of a straight line, with slope  $= a$  and  $Y$  – intercept  $= b$ .

#### Examples:

- (1) Suppose  $f$  is a linear function with slope of 2 and  $f(4) = 8$ , find  $f(x)$ .

#### Solution:

The linear function is of the form:

$$f(x) = ax + b$$

Here,  $a = 2$  and

$$f(4) = 8 = 2(4) + b$$

$$8 = 8 + b$$

$$b = 8 - 8 = 0$$

$$\text{Thus, } f(x) = 2x + b$$

$$= 2x + 0$$

$$= 2x$$

$$f(x) = 2x$$

- (2) In testing an experimental diet for hens, it was determined that the average live weight,  $w$ , (in kgs.) of a hen was statistically a linear function of the number of days,  $d$ , after the diet was begun,

where  $0 \leq d \leq 50$ . Suppose the average weight of a hen beginning the diet was 4kg. And 25 days later, it was 7kg.

- Determine  $w$  as a linear function of  $d$ .
- What is the average weight of a hen for a 10 days period?

### Solutions

- The required function is of the form:  
 $W = f(d) = md + b$ ,

where  $m$  is the slope and  $b$  is the constant intercept.

By definition,  $m = \frac{\Delta W}{\Delta d} = \frac{W_2 - W_1}{d_2 - d_1}$

From the given information:

when  $d = 0$ ;  $w = 4$  and when  $d = 25$ ;  $w = 7$ .

It follows that,  $W_1 = 4$ ;  $w_2 = 7$ ,  $d_1 = 0$ ,  $d_2 = 25$

$$m = \frac{7 - 4}{25 - 0} = \frac{3}{25}$$

Using the so-called point-slope form of a linear function:

$w - w_1 = m(d - d_1)$ , we get:

$$w - 4 = \frac{3}{25}(d - 0)$$

$$w - 4 = 0.12d$$

$$w = 0.12d + 4$$

Thus, the required linear function is

$$W = 0.12d + 4$$

- When  $d = 10$ ,  $w = 0.12(10) + 4$   
 $= 5.2$

Thus, the average weight of a hen for a 10 days period is 5.2 kg.

### 3.3 Linear Inequalities

An Inequality is simply a statement that two numbers are not equal. A **linear inequality** in the variable,  $X$ , is an inequality that can be written in the form:

$$aX + b < 0 \text{ or } aX + b \leq 0; \quad (a \neq 0).$$

### Examples

Solve the inequalities:

$$(1) \quad 2(X - 3) < 4$$

$$(2) \quad 3 - 2X \leq 6$$

### Solutions

$$(1) \quad 2(X - 3) < 4$$

$$2X - 6 < 4$$

$$2X < 4 + 6$$

$$2X < 10$$

$$\underline{2X} < \underline{10}$$

$$\frac{2}{2} < \frac{10}{2}$$

$$X < 5$$

$$(2) \quad 3 - 2X \leq 6$$

$$-2X + 3 \leq 6$$

$$-2X \leq 6 - 3$$

$$-2X \leq 3$$

$$\underline{-2X} \leq \underline{3}$$

$$\frac{-2}{-2} \leq \frac{3}{-2}$$

$$X \geq -3/2$$

The reverse in inequality sign is due to the negative effect.

### Application:

- The current ratio of any business organisation is the ratio of its current assets to its current liabilities. The Managing Director of ACE Equipment Co. has decided to obtain a short-term loan to build up inventory. The company has current assets of N350,000 and current liabilities of N80,000. How much can the Managing Director borrow if the company's current ratio must be not less than 2.5?

**Note** that funds received are considered current assets and loans are considered current liability.

### Solution

Let  $X$  = the amount to be borrowed

Then, current assets = 350,000 +  $X$

Current liability = 80,000 +  $X$

By definition,

$$\begin{aligned}\text{Current ratio} &= \frac{\text{Current Assets}}{\text{Current Liabilities}} \\ &= \frac{350,000 + X}{80,000 + X}\end{aligned}$$

Thus, according to the specification:

$$\frac{350,000 + X}{80,000 + X} \geq 2.5$$

Solving, we get:

$$\begin{aligned}350,000 + X &\geq 2.5 (80,000 + X) \\ 350,000 + X &\geq 200,000 + 2.5X \\ X - 2.5 X &\geq 200,000 - 350,000 \\ -1.5 X &\geq -150,000 \\ \underline{-1.5 X} &\geq \underline{-150,000} \\ -1.5 &\quad -1.5 \\ X &\leq 100,000\end{aligned}$$

Thus, the Managing Director can borrow not more than or as much as N100, 000 and still maintain a current ratio of not less than 2.5.

2. A publishing company finds that the cost of publishing each copy of a magazine is N0.38. The revenue from dealers of the magazine is N0.35 per copy. The advertising revenue is 10% of the revenue received from dealers for all copies sold beyond 10,000 units. What is the least number of copies which must be sold so as to have a positive profit?

### **Solution**

Let  $x$  = number of copies to be sold.

By definition,

$$\text{Profit } (\Pi) = R - C$$

Where  $R$  = Revenue from dealers + Revenue from adverts.

$$\text{Revenue from dealers} = N0.35x$$

$$\text{Revenue from adverts} = 0.10 [0.35(x - 10,000)]$$

$$\text{Total Cost } (C) = 0.38x$$

Thus,

$$\text{Profit } (\Pi) = 0.35x + 0.10[0.35(x-10,000)] - 0.38x > 0$$

Solving, we get:

$$0.35x + 0.10 [0.35x - 3500] - 0.38x > 0$$

$$0.35x + 0.035x - 350 - 0.38x > 0$$

$$0.35x + 0.035x - 0.38x > 350$$

$$0.005x > 350$$

$$\frac{0.005}{0.005} x > \frac{350}{0.005}$$

$$x > 70,000$$

Thus, the total number of copies to be sold must be greater than 70,000. That is, at least 70,001 copies.

### SELF ASSESSMENT EXERCISE

1. Solve the inequality:  $\frac{9Y + 1}{4} \leq 2Y - 1$
2. Using inequality symbols, symbolise the statement: The number of man-hours, X, to produce a commodity is not less than 2.5 nor more than 4.

## 4.0 CONCLUSION

This unit has focused on simultaneous equations, linear functions and linear inequalities as extension to the most basic mathematical tools of quantitative analysis. The unit looked at these subjects with some emphasis on the practical business applications.

## 5.0 SUMMARY

1. A simultaneous equation system is a set of equations with two or more unknown. The mostly used method of solving a simultaneous equation system is the so-called “Addition-Subtraction” or “Elimination” method.
2. A function, f, is said to be a linear function if f(x) can be written in a form:

$$f(x) = ax + b; a \neq 0$$

The function:  $Y = f(x) = ax + b$  is an equation of a straight line, with slope = a and Y – intercept = b.



3. An inequality is simply a statement that two numbers are not equal. A **linear inequality** in the variable, X, is an inequality that can be written in the form:

$$aX + b < 0 \text{ or } aX + b \leq 0; \quad (a \neq 0).$$

## 6.0 TUTOR-MARKED ASSIGNMENT

Let  $P = 0.09q + 50$  be the supply equation for a manufacturer. The demand equation for his product is:

$$P = 0.07q + 65$$

- a) If a tax of N1.50/unit is to be imposed on the manufacturer, how will the original equilibrium price be effected if the demand remains the same?
- b) Determine the total revenue obtained by the manufacturer at the equilibrium point, both before and after the tax.

## 7.0 REFERENCES/FURTHER READING

Haessuler, E. F. and Paul, R. S. (1976). *Introductory Mathematical Analysis for Students of Business and Economics*, (2<sup>nd</sup> edition.) Reston Virginia: Reston Publishing Company.

## UNIT 4    MAHEMATICAL TOOLS II: INTRODUCTION TO MATRIX ALGEBRA

### CONTENTS

- 1.0 Introduction.
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Equality of Matrices
  - 3.2 Matrix Addition
  - 3.3 Scalar Multiplication
  - 3.4 Matrix Multiplication
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

### 1.0 INTRODUCTION

An understanding of matrices and their operations is essential for input output analysis in business and economic decisions. It is also essential for solving complicated problems in simultaneous equation systems. In this unit, you will just be introduced to the basic rudimentary of matrices, with some simple applications.

A matrix is a rectangular array of numbers, called entries. Examples are:

$$\text{Matrix A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{Matrix B} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\text{Matrix C} = [1 \ 5 \ 3] = \text{row matrix}$$

$$\text{Matrix D} = \begin{bmatrix} 7 \\ 0 \\ 9 \end{bmatrix} = \text{column matrix}$$

### 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state the basic principles of matrix algebra
- analyse the important mathematical operations of matrices
- apply matrices in business decisions.

### 3.0 MAIN CONTENT

#### 3.1 Equality of Matrices

Two matrices, A and B, are said to be equal,  $A = B$ , if they have the same dimension and their corresponding entries are equal.

**Example:**

$$\text{Given } A = \begin{bmatrix} X & Y + 1 \\ 2Z & 5W \end{bmatrix} \quad B = \begin{bmatrix} 2 & 7 \\ 4 & 2 \end{bmatrix}$$

For  $A = B$ ,  $X = 2$ ;  $Y + 1 = 7$ ;  $2Z = 4$ ;  $5w = 2$

#### 3.2 Matrix Addition

If A and B are two matrices with the same dimension, then the sum,  $A + B$  is the matrix obtained by adding the corresponding entries in A and B.

**Example**

Consider an automobile dealer who sells two brands, Peugeot and Mercedes. Each is available in two colours, Red and Blue. Suppose the dealer's sales for January and February are represented respectively by the sales matrices:

$$\begin{array}{c} \text{January (J)} \\ \text{February (F)} \end{array} = \begin{array}{c} \text{Red} \\ \text{Blue} \end{array} \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} + \begin{array}{c} \text{Peugeot} \\ \text{Mercedes} \end{array} \begin{bmatrix} 1 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 7 \\ 3 & 2 & 2 & 3 \\ 4 & 2 & 4 & 2 \end{bmatrix}$$

The total sales for each brand and colour for the two months, J and F, is obtained by adding the corresponding entries in the matrices, J and F:

$$J + F = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 7 & 7 \end{bmatrix}$$

#### 3.3 Scalar Multiplication

If A is an  $(m \times n)$  matrix and K is a real number (or scalar), then  $KA$  is a scalar multiplication.

**Example**

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix} \text{ and } K = 10$$

$$\text{then } KA = 10 \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 40 & -20 \end{bmatrix}$$

**3.4 Matrix Multiplication**

If A is an (m x n) matrix and B is an (n x p) matrix, the product  $AB = C$  is of dimension (m x p). For this product to exist, the number of columns in A (that is, n) must equal the number of rows in B (that is, n). It follows that if:

$$A = (2 \times 3) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad B = (3 \times 3) \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\text{then, } AB = C = (2 \times 3) \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{bmatrix}$$

$$\begin{aligned} \text{Where: } C_{11} &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ C_{12} &= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ C_{13} &= a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ C_{21} &= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \\ C_{22} &= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ C_{23} &= a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{aligned}$$

**Examples**

1. Suppose the prices in N per unit for products A, B, and C are represented by the price matrix:

$$P = \begin{array}{c} \text{Price of} \\ \begin{matrix} A & B & C \\ [2 & 3 & 4] \end{matrix} \end{array} \\ (1 \times 3)$$

The quantities purchased are given by the quantity matrix:

$$Q = \begin{array}{c} \begin{matrix} 7 \\ 5 \\ 11 \end{matrix} \begin{matrix} \text{units of A} \\ \text{units of B} \\ \text{units of C} \end{matrix} \\ (3 \times 1) \end{array}$$

Compute the total expenditure on the products.

**Solution**

Required to compute PQ:

$$\begin{aligned}
 PQ &= [2 \quad 3 \quad 4] \begin{bmatrix} 7 \\ 5 \\ 11 \end{bmatrix} \\
 &= 2(7) + 3(5) + 4(11) \\
 &= [73] \\
 &\quad (1 \times 1)
 \end{aligned}$$

Thus, the total expenditure on products A,B, and C is N73.

2. A manufacturer of calculators has an East Coast and a West Coast plant, each of which produces Business Calculators and Standard Calculators. The manufacturing time requirements (in hours per calculator) and the assembly and packaging costs (in naira per hour) are given by the following matrices:

$$\begin{array}{c}
 \text{Hours per Unit} \\
 \begin{array}{cc}
 \text{Assembly} & \text{Packaging} \\
 T = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix} \begin{array}{l} \text{Business Calculators} \\ \text{Standard Calculators} \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Naira per Hour} \\
 \begin{array}{cc}
 \text{East Coast} & \text{West Coast} \\
 C = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix} \begin{array}{l} \text{Assembly} \\ \text{Packaging} \end{array}
 \end{array}
 \end{array}$$

- What is the total cost of manufacturing a business calculator on the East Coast?
- Calculate the manufacturer's total cost of producing the calculators in the two plant locations.

**Solutions**

Question (a) and (b) requires the product matrix:

$$TC = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix} = D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

$$\begin{array}{l}
 \text{Where} \quad D_{11} = 0.2(5) + 0.1(4) = 1.4 \\
 \quad \quad \quad D_{12} = 0.2(6) + 0.1(5) = 1.7 \\
 \quad \quad \quad D_{21} = 0.3(5) + 0.1(4) = 1.9 \\
 \quad \quad \quad D_{22} = 0.3(6) + 0.1(5) = 2.3
 \end{array}$$

It follows that the cost matrix:

East Coast	West Coast	
$\left[ \begin{array}{c} 1.4 \\ 1.9 \end{array} \right]$	$\left[ \begin{array}{c} 1.7 \\ 2.3 \end{array} \right]$	Business Calculator Standard Calculator

- a) The total cost of manufacturing a Business Calculator in the East Coast is the entry in  $D_{11} = N1.40$ .
- b) The cost matrix,  $D$ , indicates that the total cost of producing 1 unit of Business Calculator is N1.40 in the East Coast, and N1.70 in the West Coast. The cost of producing 1 unit of Standard Calculator is N1.90 in the East Coast, and N2.30 in the West Coast.

### SELF ASSESSMENT EXERCISE 1

A square matrix,  $M$ , of dimension,  $(3 \times 3)$ , has elements  $M_{ij} = 3j$  Where  $i$  represents row and  $j$  represents column. Write out the matrix.

## 4.0 CONCLUSION

This unit had exposed you to the basic principles of matrices. Of most importance are the operations in matrices, similar to mathematical operations, including:

- (i) equality of matrices;
- (ii) addition of matrices;
- (iii) scalar multiplication; and,
- (iv) matrix multiplication.

The unit also made some simple applications of such matrix operations.

## 5.0 SUMMARY

You had learned the followings from the unit's discussions:

1. Two matrices,  $A$  and  $B$ , are said to be equal,  $A = B$ , if they have the same dimension and their corresponding entries are equal.
2. If  $A$  and  $B$  are two matrices with the same dimension, then the sum,  $A + B$  is the matrix obtained by adding the corresponding entries in  $A$  and  $B$ .
3. If  $A$  is an  $(m \times n)$  matrix and  $K$  is a real number (or scalar), then  $KA$  is a scalar multiplication.
4. If  $A$  is an  $(m \times n)$  matrix and  $B$  is an  $(n \times p)$  matrix, the product

$AB = C$  is of dimension  $(m \times p)$ . For this product to exist, the number of columns in  $A$  (that is,  $n$ ) must equal the number of rows in  $B$  (that is,  $n$ ).

## 6.0 TUTOR-MARKED ASSIGNMENT

Suppose a building contractor has accepted orders for five ranch style houses, seven cape cod-style houses and twelve colonial-style houses. These orders can be represented by the row matrix

$$Q = [5 \quad 7 \quad 12]$$

Furthermore, suppose the raw materials that go into each type of house are steel, wood, glass, paint, and labour. The entries in the matrix  $R$  below give the number of units of each raw material going into each type of house:

$$R = \begin{array}{l} \text{Ranch} \\ \text{Cape Cod} \\ \text{Colonial} \end{array} \begin{array}{ccccc} \text{Steel} & \text{Wood} & \text{Glass} & \text{Paint} & \text{Labour} \\ \left[ \begin{array}{ccccc} 5 & 20 & 16 & 7 & 17 \\ 7 & 18 & 12 & 9 & 21 \\ 6 & 25 & 8 & 5 & 31 \end{array} \right] \end{array}$$

- Compute the product,  $QR$ , the amount of each raw material needed for the contract.
- Suppose that steel costs N15/unit, wood costs N8/unit, glass, paint and labour cost N5, N1, and N10 per unit respectively. These costs are represented in the column matrix:

$$C = \begin{bmatrix} 15 \\ 8 \\ 5 \\ 1 \\ 10 \end{bmatrix}$$

Compute the product  $RC$ , that is, the cost of each type of house.

## 7.0 REFERENCES/FURTHER READING

Haessler, E. F. and Paul, R. S. (1976), *Introductory Mathematical Analysis for Students of Business and Economics*, 2<sup>nd</sup> edition. Reston Virginia: Reston Publishing Company.

## UNIT 5 MATHEMATICAL TOOL III: APPLIED DIFFERENTIAL CALCULUS

### CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Derivatives and Rules of Differentiation
    - 3.1.1 Derivatives
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  - 3.2 Applications of Derivatives.
    - 3.2.1 The Marginal Cost
  - 3.3 Applied Maxima and Minima
- 4.0 Conclusion
- 5.0 Summary
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### 1.0 INTRODUCTION

This unit provides some foundations in topics of calculus that are relevant to students in business and economic decisions. You need to note that the ideas involved in calculus differ from those of algebra and geometry. One of the major problems dealt with in calculus is that of finding the rate of change, of decision variables, such as the rate of change of profit over time, the rate of change of revenue with respect to changes in product prices, and the like.

### 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define ‘derivate’ of function
- apply the techniques of finding the derivatives by proper application of rules
- differentiate between the “maxima” and “minima” principles
- apply calculus in business and economic decisions.

### 3.0 MAIN CONTENT

#### 3.1 Derivatives and Rules of Differentiation

##### 3.1.1 Derivatives

For a given function,  $f$ , the derivative of  $f$  at  $X_0$ , denoted by  $f'(X_0)$ , is the limit:



$$f'(X_0) = \lim_{h \rightarrow 0} \frac{f(X_0 + h) - f(X_0)}{h}$$

If a given function,  $f$ , has a derivative at  $X_0$ , we say that  $f$  is differentiable at  $X_0$ . To find the derivative of  $f$  is to differentiate  $f$ , and the process of finding a derivative is called differentiation.

If a function,  $f(X)$ , has a derivative of  $f'(X_0)$  at  $X_0$ , then:

- (a) the instantaneous rate of change of  $f(X)$  relative to  $X$  at  $X_0$  is  $f'(X_0)$
- (b) the slope of the curve (or the slope of the tangent line to the curve),

$$Y = f(X) \text{ at } (X_0, f(X_0)) \text{ is } f'(X_0)$$

In general, for a given function  $f$ , the derivative of  $f$  is the function of  $f'(X)$  given by:

$$f'(X) = \lim_{h \rightarrow 0} \frac{f(X + h) - f(X)}{h}$$

We can refer to the derivative  $f'(X)$  as a function whose output value at  $X_0$  is the slope of the curve,  $Y = f(X)$  at  $(X_0, f(X_0))$  or the instantaneous rate of change in  $f(X)$  at  $X = X_0$ .

### Example

Let  $f$  be defined by  $f(X) = X^2$ . Then, the derivative is:

$$\begin{aligned} f'(X) &= \lim_{h \rightarrow 0} \frac{f(X + h) - f(X)}{h} = \lim_{h \rightarrow 0} \frac{(X + h)^2 - X^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{X^2 + 2Xh + h^2 - X^2}{h} \\ &= \lim_{h \rightarrow 0} (2X + h) \\ &= 2X \text{ as } h \rightarrow 0 \end{aligned}$$

The symbol,  $dy/dx$ , is used to represent the derivative of a function,  $Y = f(X)$ . Thus, if  $Y = f(X)$ , where  $f$  is differentiable, then

$$\frac{dy}{dx} = f'(X)$$

### 3.1.2 Rules of Differentiation

The rules of differentiation give us some efficient shortcuts for calculating derivatives. It is, therefore, not necessary to use the limit procedures, as in the example above.

The rules are as follows:

**Rule 1 (Constant Rule):** This states that the derivative of a constant function is zero. Thus, if  $Y = f(X) = C$ , where  $C$  is any constant number, then

$$dy/dx = f'(X) = 0$$

**Rule 2 (Constant Multiplier Rule):** If the function  $f$ , is differentiable at  $X$ , then so is any constant multiple of  $f$ ;

$$\frac{d}{dx} [cF(X)] = cF'(X)$$

**Rule 3 (The Power Rule):** For any positive integer,  $n$ , the function  $f(X) = X^n$  is differentiable everywhere, and  $\frac{d}{dx} (X^n) = nX^{n-1}$

**Rule 4 (Sum and Difference Rule):** If  $f(X)$  and  $g(X)$  are both differentiable at  $X$ , then so are  $f(X) + g(X)$  and  $f(X) - g(X)$ . And

$$\frac{d}{dx} [f(X) \pm g(X)] = \frac{d}{dx} [f(X)] \pm \frac{d}{dx} [g(X)]$$

#### Examples

$$1. \quad \frac{d}{dx} (X^5 + X^4) = \frac{d}{dx} (X^5) + \frac{d}{dx} (X^4) = 5X^4 + 4X^3$$

$$2. \quad \frac{d}{dx} (1 - X^3) = \frac{d}{dx} (1) - \frac{d}{dx} (X^3) = 0 + 3X^2 = 3X^2$$

**Rule 5 (The Product Rule):** If the functions  $f(X)$  and  $g(X)$  are both differentiable at  $X$ , then so is their product,  $f(X)g(X)$ . And,

$$\frac{d}{dx} [f(X) \cdot g(X)] = f(X) \frac{d}{dx} [g(X)] + g(X) \frac{d}{dx} [f(X)]$$

#### Example

$$\frac{d}{dx} [(X^5 + 2X^3 - 5)(3X^4 - 2X^2 + 8)]$$

$$\begin{aligned}
 &= (X^5 + 2X^3 - 5) \frac{d}{dx} [3X^4 - 2X^2 + 8] + (3X^4 - 2X^2 + 8) \frac{d}{dx} [X^5 + 2X^3 - 5] \\
 &= (X^5 + 2X^3 - 5)(12X^3 - 4X) + (3X^4 - 2X^2 + 8)(5X^4 + 6X^2)
 \end{aligned}$$

**Rule 6 (The Quotient Rule):** If the functions  $f(X)$  and  $g(X)$  are differentiable at  $X$ , then so is their quotient,  $f(X)/g(X)$ , provided that  $g(X) \neq 0$ . And,

$$\begin{aligned}
 \frac{d}{dx} \left[ \frac{f(X)}{g(X)} \right] &= \frac{g(X) \frac{d}{dx} [f(X)] - f(X) \frac{d}{dx} [g(X)]}{[g(X)]^2} \\
 &= \frac{g(X)f'(X) - f(X)g'(X)}{[g(X)]^2}
 \end{aligned}$$

### Example

If  $Y = \frac{3X^2 - 1}{X + 2}$  then by the quotient rule:

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(X + 2) \frac{d}{dx} (3X^2 - 1) - (3X^2 - 1) \frac{d}{dx} (X + 2)}{(X + 2)^2} \\
 &= \frac{(X + 2)(6X) - (3X^2 - 1)(1)}{(X + 2)^2}
 \end{aligned}$$

**Rule 7 (The Chain Rule):** If  $Y = f(U)$  is a differentiable function of  $U$ , and  $U = g(X)$  is a differentiable function of  $X$ , then the composite function,  $Y = f(g(X))$  is a differentiable function of  $X$ . And,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

### Example

Find  $d/dx(2X^3 - 5X^2)^{78}$

### Solution

Let  $Y = f(U)$  and  $U = g(X) = 2X^3 - 5X^2$ , then  $Y = U^{78}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\text{where } \frac{dy}{du} = 78U^{77}; \frac{du}{dx} = 6X^2 - 10x$$

$$\text{Thus } \frac{dy}{dx} = 78U^{77}(6X^2 - 10x)$$

$$= 78(2X^3 - 5X^2)^{77} (6X^2 - 10X)$$

(Since  $U = 2X^3 - 5X^2$ )

### 3.2 Applications of Derivatives

The following few examples demonstrate the economic applications of the concept of derivatives.

1. Find the rate of change of the variable  $Y = X^4$  with respect to  $X$ . Evaluate the rate when  $X = 2$  and when  $X = -1$ .

#### Solution

By the concepts of derivatives, if  $Y = X^4$ ,

$$\frac{dy}{dx} = 4x^3$$

$$\text{When } X = 2, \frac{dy}{dx} 4(2)^3 = 32$$

This implies that if  $X$  increases by a small amount, then  $Y$  will increase approximately 32 times as much. Or put differently,  $Y$  increases 32 times as fast as  $X$  does.

$$\text{When } X = -1, \frac{dy}{dx} 4(-1)^3 = -4$$

Implying that  $Y$  decreases 4 times as fast as  $X$  increases.

2. Let  $P = 100 - q^2$  represent a demand function. Find the rate of change of price,  $P$ , with respect to unit changes in  $q$ . How fast is the price changing with respect to  $q$ , when  $q = 5$ , assuming that  $P$  is in naira?

#### Solution

The rate of change of  $P$  with respect to  $q$  is:

$$\frac{dP}{dq} = \frac{d}{dq} (100 - q^2) = -2q$$

when  $q = 5$ ,  $\frac{dP}{dq} = -2(5) = -10$

This implies that when 5 units are demanded, an increase of one extra unit of demand will correspond to a decrease of approximately N10 in the price per unit that consumers are willing to pay.

### 3.2.1 The Marginal Cost

A manufacturer's total cost function,  $C = f(q)$ , gives the total cost of producing and marketing  $q$  units of a product. The rate at which costs,  $C$ , changes with respect to output,  $q$ , is referred to as the marginal cost. Thus, by definition:

$$\text{Marginal Cost (MC)} = \frac{dC}{dq}$$

#### Example

Suppose  $C = f(q) = 0.1q^2 + 3$  is a cost function, where  $C$  is in naira and  $q$  is in kgs.

Then, the marginal cost of production would be:

$$\begin{aligned} \text{MC} &= \frac{dC}{dq} = \frac{d}{dq} [0.1q^2 + 3] \\ &= 0.2q \end{aligned}$$

The marginal cost when 4kg are produced, for example, is

$$\frac{dC}{dq} \text{ given that } q = 4 = 0.2(4) = 0.80$$

This means that the additional cost of producing an additional kg. of output is N0.80.

### 3.2.2 The Marginal Revenue (MR)

Suppose  $R = f(q)$  is the Total Revenue Function of a manufacturer.

$$\text{Then MR} = \frac{dR}{dq}$$

### Example

Suppose a manufacturer sells a product at N2 per unit. If  $q$  units are sold, the total revenue:

$$R = pq = 2q$$

$$\text{Marginal Revenue (MR)} = \frac{dR}{dq} = P = 2 = \text{N2}$$

This implies that the revenue is changing at the rate of N2 per unit, regardless of the number of units sold.

### 3.3 Applied Maxima and Minima

The concept of derivatives can be used in solving maximisation and minimisation problems. To maximise or minimise an objective function all we need is to differentiate the function, set it equal to zero and solve for either the maximum or minimum values, depending on the objective function. This process of differentiating the objective function and setting the result equal to zero is referred to as the First – Order Condition (FOC) for maximisation and minimisation.

To confirm maximum or minimum conditions, we take the second derivative of the objective function. By evaluation, if the second derivative exceeds zero, then we will be at the minimum point, and the values at this point will be the minimum values. On the other hand, if the second derivative is less than zero, we will be at the maximum point, with the associated values being the maximum values.

The process of taking the second derivative and evaluating is called the Second – Order Condition (SOC).

These concepts are illustrated in the following examples.

1. A manufacturer's cost function is given by:

$$C = \frac{q^2}{4} + 3q + 400 = 0.25q^2 + 3q + 400$$

where  $q$  = number of units produced.

At what level of output will average cost per unit be a minimum? What is the minimum cost?

### Solution

By definition, the Average Cost (AC) function is given by:

$$\begin{aligned} AC &= \bar{C} = C/q \\ &= \frac{0.25q^2 + 3q + 400}{q} \\ &= 0.25q + 3 + 400q^{-1} \end{aligned}$$

To minimise the Average Cost, we differentiate to get:

$$AC = \frac{d\bar{C}}{dq} = 0.25 - 1(400)q^{-1}$$

Setting this equal to zero according to the FOC, we get:

$$\begin{aligned} 0.25 - 400q^{-2} &= 0 \\ &= \frac{400}{q^2} \\ 0.25q^2 &= 400 \\ q^2 &= \frac{400}{0.25} = 1600 \\ q &= \pm \sqrt{1600} = \pm 40 \end{aligned}$$

For  $q > 0$ , the cost minimising level of output is  $q = 40$  units. Thus, the level of output for which average cost per unit will be minimum is 40 units.

2. The demand equation for a manufacturer's product is given by:

$$P = \frac{80 - X}{4}$$

- (a) At what value of  $X$  will there be a maximum revenue?
- (b) What is the maximum revenue?

### Solution

By definition, revenue ( $R$ ) =  $PX$

$$\begin{aligned} \text{Thus, } R &= PX = (20 - 0.25X)X \\ &= 20X - 0.25X^2 \end{aligned}$$

By the first - order condition:

$$\frac{dR}{dX} = 20 - 0.5X$$

Setting this equal to zero and solving for  $X$ , we get:

$$20 - 0.5X = 0$$

$$0.5X = 20$$

$$X = \frac{20}{0.5} = 40$$

Thus, the value of X which the revenue will be maximised is 40 units.

### SELF ASSESSMENT EXERCISE 1

A Computer Service Bureau generates revenue at the rate  $R(t) = 20 + 16t - t^2$  thousand naira in its  $t$  years of operation, while its average cost of operation is

$$\bar{c}(t) = 2t - \frac{65}{t} - 20 \text{ thousand naira}$$

- When is the revenue of the firm greatest and what is its value?
- For how many years is the operation of the firm profitable?
- When is the profit of the firm optimum?

## 4.0 CONCLUSION

This unit has provided you with some foundations in topics of calculus that are relevant to students in business and economic decisions. You need to note that the ideas involved in calculus differ from those of algebra and geometry. You should also note that the major problems dealt with in calculus is that of finding the rate of change of decision variables, such as the rate of change of profit over time, the rate of change of revenue with respect to changes in product prices, and the like.

You observed from this unit that the concept of derivatives can be used in solving maximisation and minimisation problems. To maximise or minimise an objective function all we need is to differentiate the function, set it equal to zero and solve for either the maximum or minimum values, depending on the objective function. This process of differentiating the objective function and setting the result equal to zero is referred to as the First – Order Condition (FOC) for maximisation and minimisation.

To confirm maximum or minimum conditions, you take the second derivative of the objective function. By evaluation, if the second derivative exceeds zero, then we will be at the minimum point, and the values at this point will be the minimum values. On the other hand, if the second derivative is less than zero, we will be at the maximum point, with the associated values being the maximum values.



The process of taking the second derivative and evaluating is called the Second – Order Condition (SOC).

## 5.0 SUMMARY

In general, for a given function  $f$ , the derivative of  $f$  is the function of  $f'$  ( $X$ ) given by:

$$f'(X) = \lim_{h \rightarrow 0} \frac{f(X + h) - f(X)}{h}$$

We can refer to the derivative  $f'(X)$  as a function whose output value at  $X_0$  is the slope of the curve,  $Y = f(X)$  at  $(X_0, f(X_0))$  or the instantaneous rate of change in  $f(X)$  at  $X = X_0$ .

The rules of differentiation give us some efficient shortcuts for calculating derivatives. It is, therefore, not necessary to use the limit procedures, as in the example above.

The rules are as follows:

A manufacturer's total cost function,  $C = f(q)$ , gives the total cost of producing and marketing  $q$  units of a product. The rate at which costs,  $C$ , changes with respect to output,  $q$ , is referred to as the marginal cost. Thus, by definition:

$$\text{Marginal Cost (MC)} = \frac{dC}{dq}$$

## 6.0 TUTOR-MARKED ASSIGNMENT

A manufacturer has found that if his product is priced at N90/unit then his weekly demand is 50 units, but the demand rises to 60 units per week if the selling price is N70/unit. His weekly fixed cost is N3, 000 and variable cost N20/unit.

Find the level of production which maximises profit and determine the maximum profit.

## 7.0 REFERENCES/FURTHER READING

Haessler, E. F. and Paul, R. S. (1980). *Calculus for the Managerial, Life, and Social Sciences*. Reston Virginia: Reston Publishing Inc.

**MODULE 2**

Unit 1	Statistical Tool I: Measures of Averages
Unit 2	Statistical Tools II: Measures of Variability or Dispersion
Unit 3	Statistical Tools III: Sets and Set Operations
Unit 4	Statistical Tools IV: Probability Theory and Applications
Unit 5	Correlation Theory

**UNIT 1      STATISTICAL TOOLS I: MEASURES OF  
AVERAGES**

1.0	Introduction
2.0	Objectives
3.0	Main Content
3.1	Arithmetic Mean
3.1.1	<i>Arithmetic Mean for Observations of Equal Weight</i>
3.1.2	<i>Weighted Arithmetic Mean</i>
3.1.3	Arithmetic Mean for a Grouped Data
3.2	The Median
3.2.1	Median for a Grouped Data
3.3	The Mode
3.3.1	<i>Mode for a Grouped Data</i>
4.0	Conclusion
5.0	Summary
6.0	Tutor-Marked Assignment
7.0	References/Further Reading

**1.0      INTRODUCTION**

Units 1 and 2 will be devoted mainly to the simple statistical tools needed for business and economic decisions. In this unit, we focus on the concept of averages. It is essential for any business person to understand this concept as it forms the basis for the establishment of targets and yardsticks for measures of success. The units introduce two important and often used statistical measures in quantitative techniques:

- the measure of central tendency
- the measure of disper.

## **2.0 OBJECTIVES**

At the end of this unit, you should be able to:

- state the importance and measures of central tendency or averages
- state the measures and uses of dispersion or deviations from averages
- apply the principles of averages and dispersion in day-to-day business decisions.

## **3.0 MAIN CONTENT**

### **3.1 Measures of Central Tendency or Averages**

The three important measures of central tendency to be discussed here, among others include:

1. arithmetic mean
2. median
3. mode

### 3.2 Arithmetic Mean

The arithmetic mean is used in estimating the average of a given set of observations. It is the most accurate measure of an average. Of major concern in the present discussion are three levels of estimates:

- arithmetic mean of observations of equal weights or levels of importance.
- arithmetic mean of weighted set of observations (weighted arithmetic mean)
- arithmetic mean for a grouped data (or set of observations)

#### 3.2.1 Arithmetic Mean for Observations of Equal Weight

The computation of arithmetic mean for a set of observations of equal weight or of equal levels of importance follows the following process:

Let  $X$  represent a variable of which the observations that are of equal weights are given as

$$X_i \text{ (} i= 1,2,3,\dots,n\text{)}.$$

Then, the arithmetic mean of this set of observations is defined by:

$$\bar{X} = \frac{\sum x_i}{n} = \frac{X_1 + X_2 + X_3 + \dots + X_n}{n}$$

Where  $n$  = number of observations

#### Example

Let  $X_i = 2, 2, 4, 7, 3, 1$

$$\bar{X} = \frac{\sum x_i}{n} = \frac{1 + 2 + 2 + 3 + 4 + 7}{6} = \frac{19}{6} = 3.1667$$

Notice that the set of observations were presented in an array before the computation of the mean. It is therefore advisable to always array a set of observations before processing them for subsequent analysis. By an “array”, we mean the presentation of observations in either an ascending or descending order.

#### 3.1.2 Weighted Arithmetic Mean

When a set of observations are of different weights or levels of importance, the computation of the arithmetic mean takes into account the individual weights. By definition, Let  $X$  represent a variable, with observations of different weights, and

Let  $X_i(i=1,2,3,\dots,n)$  = individual observations on the variable  $X$   
 $w_i(i=1,2,3,\dots,n)$  = weights attached to individual observation  
 $\bar{X}_w$  = **weighted mean of the observations on  $X$**

$$\text{Then, } \bar{X} = \frac{\sum w_i x_i}{\sum w_i} = \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + \dots + w_n x_n}{w_1 + w_2 + w_3 + \dots + w_n}$$

### Example

Assume three levels of examinations, Quiz, Mid-semester, and Final examination. The weights assigned to each of these examinations and scores for a particular participant is as summarized below:

Exam	Weight( $w_i$ )	Score( $x_i$ )
Quiz	1	65
Mid-Semester	3	50
Final Exam	6	45

What is the average score of the participant for the three examinations?

### Solution

Using our definition,

$$\bar{X}_w = \frac{\sum w_i x_i}{\sum w_i}$$

We compute as follows:

Exam	Weight ( $w_i$ )	Score ( $x_i$ )	$w_i x_i$
Quiz	1	65	65
Mid-Semester	3	50	150
Final Exam	6	45	270
	<b><math>\sum w_i=10</math></b>		<b><math>\sum w_i x_i=485</math></b>

$$\bar{X}_w = \frac{\sum w_i x_i}{\sum w_i} = \frac{485}{10} = 48.5$$

The required average score is 48.5

### 3.1.3 Arithmetic Mean for a Grouped Data

When a set of observations is presented in a grouped form or in class limits or boundaries, the computation of the arithmetic mean is done according to the following definitions:

$$\bar{X}_g = \frac{\sum fx}{\sum f} = \frac{\sum fx}{n}$$

Where  $\sum f = n$

$\bar{X}_g$  = Grouped mean of the set of data on variable x.

f = frequency of observations

x = mid-value of each class limit defined by

$X = \frac{\text{Lower Class Limit} + \text{Upper Class Limit}}{2}$

for each class limit.

#### Example

The following is a data on the daily wages paid to workers in a given factory:

Daily Wages (in N)	No. of Workers (f)
200 – 300	15
301 – 401	30
402 – 502	45
503 – 603	60
Total	150

*Compute the average daily wages.*

#### Solution

We re-tabulate the data as follows:

Daily Wages	f	x	fx
-------------	---	---	----

200-300	15	250	3750
301-401	30	351	10530
402-502	45	452	20340
503-603	60	553	33180
			$\Sigma fx = 67800$

Note: Computation of the values of X (i.e., the mid-value of classes) is as follows:

$$200 + 300 = 500 = 250$$

$$\frac{2}{2} \quad \frac{2}{2}$$

$$\frac{301 + 401}{2} = \frac{702}{2} = 351$$

$$\frac{2}{2} \quad \frac{2}{2}$$

$$\frac{402 + 502}{2} = \frac{904}{2} = 452$$

$$\frac{2}{2} \quad \frac{2}{2}$$

$$\frac{503 + 603}{2} = \frac{1106}{2} = 553$$

$$\frac{2}{2} \quad \frac{2}{2}$$

### 3.2 The Median

The average of a given set of data can also be estimated using the median. The median is a measure of central tendency which appears in the “middle” of an ordered sequence of values or observations. That is, half of the observations in a set of data are lower than the median value and half are greater than it. To compute the median from a set of raw data, we must first array the data. If we have odd number of observations, the median is represented by the numerical value of the  $(n+1)/2$  arrayed observation. On the other hand, if the number of observations in the sample is an even number, the median is represented by the mean or average of the two middle values in the ordered array.

#### Example

Consider the following raw data on hourly wage rate for six executive secretaries:

Raw data (in =N=)	
$X_1 = 950$	$X_4 = 950$
$X_2 = 300$	$X_5 = 850$
$X_3 = 1000$	$X_6 = 750$

Compute the median hourly wage rate for the six secretaries.

**Solution**

### Ordered Array

$X_i = 300, 750, 850, 950, 950, 1000$

Since this involves an “even” number of observations, the median will be the average of the two middle values:

$$= \frac{850 + 950}{2} = \frac{1800}{2} = 900$$

Thus, the median hourly wage rate is N900

As another example, consider the following ordered array of  $n = 5$  students’ mid-semester examination results:

$X_i = 64, 79, 88, 90, 94$

The median score will be the  $\frac{(n+1)}{2}$ th or  $\frac{(5+1)}{2}$ th ordered observation, that is, 3<sup>rd</sup> ordered observation, which is 88, since this involves “odd” number of observations.

### 3.2.1 Median for a Grouped Data

The median value for a grouped data is computed according to the following definitions:

For a grouped data:

$$\text{Median} = L + \frac{(N/2 - f_b)c}{f_m}$$

Where  $L$  = Lower actual class limited of the median group, where the median group can be regarded as the group (or class limit) with the highest frequency.

$N = \Sigma f$  = total number of observations in the given data

$f_b$  = total frequency of all classes below the median group

$C$  = class width for the actual class limit.

$f_m$  = frequency of the median group



**Note:** Class Width = Upper Class Limit – Lower Class Limit

**Example**

Giving the following table on the share prices of a quoted company over a period of 60 days:

Price in =N=	No. of Days (f)
110-114	2
115-119	6
120-124	8
125-129	12
130-134	14
135-139	8
140-144	6
145-149	4
$\Sigma f = 60$	

Find the median share price.

*Solution*

We re-tabulate the data to get the needed computational values:

State Class Limits	Actual Class Limits	Frequency (f)
100-114	109.5-114.5	2
115-119	114.5-119.5	6
120-124	119.5-124.5	8
125-129	124.5-129.5	12
130-134	129.5-134.5	14
130-139	134.5-139.5	8
140-144	139.5-144.5	6
145-149	144.5-149.5	4
		$\Sigma f = 60$

From the re-tabulated data, the median group, that is, the group with the highest frequency (14) is (129.5-134.5).

By definition,

$$\text{Median Price} = L + \frac{(N/2 - f_b)C}{f_m}$$

Where  $L = 129.5$ ;  $N = 60$ ;  $f_b = 12+8+6+2 = 28$ ;  $C = 5$ ;  $f_m = 14$ .

$$\begin{aligned} \text{It follows that median price} &= 129.5 + \frac{(30 - 28)(5)}{14} \\ &= 129.5 + \frac{10}{14} \end{aligned}$$

$$= 129.5 + 0.71$$

$$= 130.21$$

Thus, the median share price is N130.21

### 3.3 The Mode

The mode is a quick measure of central tendency or average. The mode is the most typical or most frequently observed value in a given set of data. It is the observation with the highest frequency in a given set.

For the set of data,  $X = 1, 2, 2, 4$ , the value with the highest frequency is 2. Hence, the mode for the given data is 2.

#### 3.4.1 Mode for a Grouped Data

The computational process for the mode of a grouped data is similar to that of the median. The process is as follows:

$$\text{Mode} = L + \frac{(f_m - f_L)C}{(f_m - f_L) + (f_m - f_h)}$$

Where

- L = Lower actual class limit of the modal group
- $f_m$  = frequency of the modal group
- $f_L$  = frequency of the group before the modal group
- $f_h$  = frequency of the group after the modal group
- C = class width for the actual class limit

#### Example

Consider the following data on the sales by 90 sales representatives:

Sales (N'000s)	No. of Sales Reps.(f)
10 – 15	10
16 – 21	36
22 – 27	28
28 – 33	10
34 – 39	6

Calculate the modal value of sales.

#### Solution

Re-tabulating the data, we get:

Stated Class Limits	Actual Class Limits	Frequency (f)
10-15	9.5-15.5	10
16-21	15.5-21.5	36
22-27	21.5-27.5	28
28-33	27.5-33.5	10
34-39	33.5-39.5	6

From the re-tabulated data,  $L = 15.5$ ;  $f_m = 36$ ;  $f_L = 10$ ;  $f_h = 28$ ;  $C = 6$ .

It follows that:

$$\begin{aligned} \text{Mode} &= L + \frac{(f_m - f_L)C}{(f_m - f_L) + (f_m - f_h)} \\ &= 15.5 + \frac{(36 - 10)(6)}{(36 - 10) + (36 - 28)} \\ &= 15.5 + \frac{156}{34} \\ &= 20.09 \end{aligned}$$

Thus, the modal value of sales is  $N(20.09 \times 1000) = N20,090$  (since values are in thousands).

### SELF ASSESSMENT EXERCISE 1

Enumerate, with simple examples, three important measures of averages in a business environment.

## 4.0 CONCLUSION

To be in command of the latest management techniques, you have to be knowledgeable on basic statistical tools. You will find it difficult making reasonable and effective business projections without some backgrounds in the basic principles of statistics, ranging from measures of averages to forecasting techniques. This unit served as the starting point in the knowledge of these basic principles. It identifies the basic measures of averages including: the arithmetic mean, the median, and the mode, with some simple business applications.

## 5.0 SUMMARY

In a nutshell, this unit has exposed you to the following statistical concepts:

1. The **arithmetic mean** used in estimating the average of a given set of observations. It is the most accurate measure of an

average, and of major concern in the present discussion are three levels of estimates:

- Arithmetic Mean of observations of equal weights or levels of importance.
  - Arithmetic Mean of weighted set of observations (weighted Arithmetic Mean)
  - Arithmetic Mean for a grouped data (or set of observations)
2. The **median**, a measure of central tendency which appears in the “middle” of an ordered sequence of values or observations. That is, half of the observations in a set of data are lower than the median value and half are greater than it.
  3. The **mode** which is a quick measure of central tendency or average. The mode is the most typical or most frequently observed value in a given set of data. It is the observation with the highest frequency in a given set.

## 6.0 TUTOR-MARKED ASSIGNMENT

The award of a contract is based on a contractor’s scores on credibility test over five years in business. The scores are weighted as follows:

Year:	1	2	3	4	5
Weight	1	3	5	7	9

If the score of five of the contractors (Adamu, Okoli, Babalola, Udoh and Okoh) are as shown below:

Contractor	Year				
	1	2	3	4	5
Adamu	62	62	50	61	48
Okoli	46	56	67	50	62
Babalola	51	54	60	55	58
Udoh	63	60	49	52	61
Okoh	58	62	60	52	70

and a contract will be awarded to the contractor with the highest average score, which contractor will the contract, be given to?

## 7.0 REFERENCES/FURTHER READING

Haessler, E. F. and Paul, R. S. (1976). *Introductory Mathematical Analysis for Students of Business and Economics*, 2<sup>nd</sup> edition. Reston Virginia: Reston Publishing Company.

## **UNIT 2     STATISTICAL TOOLS II: MEASURES OF VARIABILITY OR DISPERSION**

### **CONTENTS**

- 1.0 Introduction
- 2.0 Objectives
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    - 3.4.1 Variance and Standard Deviation for a Grouped Data
  - 3.5 The Coefficient of Variation
  - 3.6 The Measures of Skewness
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### **1.0 INTRODUCTION**

The second most important characteristic which describes a set of data is the amount of variation, deviation, scatter, or spread in the data. Deviation from an established average is a signal that a problem requiring managerial solution exists. A sales person in a commercial bank who consistently performs below an established average or target, for example, signals a problem in his or her marketing activities, which requires managerial decision. In this unit, we outline five important measures of variations, including:

1. the range
2. mean deviation

3. variance
4. standard deviation
5. coefficient of variation

The variation or dispersion measures the degree of non-uniformity of observations in a given set of data. The greater the variation, the more non-uniform the observations in a given set of data is. A manager's role is to ensure a uniform set of observations as this is essential for reliability in forecasting and planning.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain the different measures of variation or deviations
- identify the best unbiased measure of variation
- recognise when a set of data is biased or unreliable
- state the managerial implications of non-uniformity in data behaviours
- recognise a decision problem needing managerial solutions.

## 3.0 MAIN CONTENT

### 3.1 The Measures of Variations or Dispersion

As noted above, for the purpose of this course, we discuss five major measures of dispersion in the behaviour of business and/or economic variables: the range, the mean deviation, the variance, the standard deviation, and the coefficient of variation. Another important related measure that is primarily concerned with distribution of some business data is the Pearson Coefficient of Skewness. You will be exposed to its computation, interpretations, and managerial implications.

#### 3.1.1 The Range

The Range (R) of a given set of ungrouped data can be determined from an ordered array as the difference between the highest observation and the lowest observation.

Let  $X_h$  = Highest observation  
 $X_L$  = Lowest observation

The range,  $R = X_h - X_L$

Given the arrayed data:  $X = 2, 5, 8, 9, 12, 13, 18$ , the range will be:  
 $R = 18 - 2 = 16$ .

### 3.2 The Mean Deviation (MD)

The Mean Deviation can be defined simply by the following relationship:

$$MD = \frac{\sum |X - \bar{X}|}{n}$$

where  $\frac{\sum |X - \bar{X}|}{n}$  = sum of the absolute values of deviation from

arithmetic mean

n = number of observation

As an example, consider again the arrayed data, X = 2,5,8,9,12,13,18.

The mean deviation, MD, can be computed as follows:

$$\bar{X} = \frac{\sum X}{n} = \frac{67}{7} = 9.57$$

By tabulation,

X	(X - $\bar{X}$ )	X - $\bar{X}$
2	-7.57	7.57
5	-2.57	2.57
8	-1.57	1.57
9	-0.57	0.57
12	2.43	2.43
13	3.43	3.43
18	8.43	8.43
$\sum  X - \bar{X}  =$		<u>26.57</u>

Thus,

$$MD = \frac{\sum |X - \bar{X}|}{n} = \frac{26.57}{7} = 3.7957$$

### 3.4 The Variance

The Variance for a given set of an ungrouped data can be defined by:

$$\text{Variance} = S^2 = \frac{\sum X^2 - (\sum X/n)^2}{n - 1}$$



where  $X$  represents the numerical values of the given set of an ungrouped data.

Continuing with our earlier example, where

$X = 2, 5, 8, 9, 12, 13, 18$ , and by tabulation:

$X$	$X^2$
2	4
5	25
8	64
9	81
12	144
13	169
18	324

$$\sum X = 67; \sum X^2 = 811;$$

$$S^2 = \frac{(\sum X)^2}{n} = \frac{(67)^2}{7} = \frac{4489}{7} = 641.29$$

$$\begin{aligned} \text{Thus, } S^2 &= \frac{\sum X^2 - (\sum X)^2/n}{n-1} = \frac{811 - 641.29}{7-1} \\ &= \frac{169.71}{6} = 28.285 \end{aligned}$$

Thus, the variance of the given set of ungrouped data is 28.285. This is often regarded as a crude and over-estimated measure of dispersion. By interpretation of this result, you can infer that, on the average, each individual observation in the given data differs from the established average of 9.57 by about 28 units. Observe that this type of data will be difficult to make forecasting with and plan with.

### 3.4 The Standard Deviation

Simply stated, the standard deviation is the most useful measure of variation. It can be defined as the square root of the variance for a given set of data.

Thus,

$$\text{Standard deviation} = S = \sqrt{S^2}$$

Or,

$$s = \sqrt{\frac{\sum X^2 - (\sum X)^2/n}{n-1}}, \text{ for ungrouped data}$$

The standard deviation for the last example is:

$$S = \sqrt{s^2} = \sqrt{28.285} = 5.318$$

Again, by interpretation, this implies that each individual observation in the given set of data deviates from the established average of 9.57 by about 5 units.

### 3.4.1 Variance and Standard Deviation for a Grouped Data

The computation of variance and standard deviation for a grouped data is illustrated by the following example.

The Variance and Standard Deviation for a grouped data are defined by the following formulations:

$$\text{Variance} = s^2 = \frac{\sum fx^2 - (\sum fx)^2/n}{n-1}$$

$$\text{Standard Deviation} = \sqrt{s^2} = \sqrt{\frac{\sum fx^2 - (\sum fx)^2/n}{n-1}}$$

#### Example

The following data presents the profit ranges of 100 firms in a given industry.

Profits (N'millions)	No. of Firms (f)
10 – 15	8
16 – 21	18
22 – 27	20
28 – 33	12
34 – 39	15
40 – 45	17
46 – 51	10
	$\sum f = n = 100$

We are required to compute the variance and standard deviation of profits within the industry.

## Solutions

By definition,

$$\text{Variance} = S^2 = \frac{\sum fx^2 - (\sum fx)^2/n}{n - 1}$$

$$\text{Standard Deviation} = \sqrt{S^2} = \sqrt{\frac{\sum fx^2 - (\sum fx)^2/n}{n - 1}}$$

The computational process is as follows:

Profits (N millions)	Frequency (f)	Mid-Value (x)	fx	x <sup>2</sup>	fx <sup>2</sup>
10-15	8	12.5	100	156.25	1250
16-21	18	18.5	333	342.25	6160.5
22-27	20	24.5	490	600.25	12005
28-33	12	30.5	366	930.25	11163
34-39	15	36.5	547.5	1332.25	19983.75
40-45	17	42.5	722.5	1806.25	30706.25
46-51	10	48.5	485	2352.25	23522.50
	<u><math>\sum f = n = 100</math></u>		<u><math>\sum fx = 3044</math></u>		<u><math>104791 = \sum fx^2</math></u>

## Summary

$$\frac{(\sum fx)^2}{n} = \frac{(3044)^2}{100} = 92659.36$$

$$\sum fx^2 = 104791$$

It follows that:

$$\begin{aligned} \text{Variance} = S^2 &= \frac{\sum fx^2 - (\sum fx)^2/n}{n - 1} = \frac{104791 - 92659.36}{100 - 1} \\ &= \frac{12131.64}{99} = 122.54 \end{aligned}$$

$$\text{Standard Deviation} = \sqrt{S^2} = \sqrt{122.54} = 11.07$$

Thus, the required variance and standard deviation are 122.54 and 11.07 respectively.

### 3.5 The Coefficient of Variation

Unlike other measures of variability, the coefficient of variation is a relative measure. It is particularly useful when comparing the variability of two or more sets of data that are expressed in different units of measurements.

The coefficient of variation measures the standard deviation relative to the mean and is computed by:

$$\text{Coefficient of Variation} = CV = \frac{[S]}{X} 100\%$$

The coefficient of variation is also useful in the comparison of two or more sets of data which are measured in the same units but differ to such an extent that a direct comparison of the respective standard deviations is not very helpful.

**As an example**, suppose a potential investor is considering the purchase of shares in one or two companies, A or B, which are listed on the Nigerian Stock Exchange (NSE). If neither company offered dividends to its shareholders and if both companies were rated equally high in terms of potential growth, the potential investor might want to consider the volatility of the two stocks to aid in the investment decision.

Now, suppose each share of stock in Company A has an average of N50 over the past months with a standard deviation of N10. In addition, suppose that in this same time period, the price per share for Company B's stock averaged N12 with a standard deviation of N4. Observe that in terms of actual standard deviations, the price of Company A's shares seems to be more volatile than that of Company B. However, since the average prices per share for the two stocks are so different, it would be more appropriate for the potential investor to consider the variability in price relative to the average price in order to examine the volatility/stability of two stocks.

The coefficient of variation of company A's stock is

$$CV_A = \frac{[S_A]}{X_A} 100\% = \frac{[N10]}{N50} 100\% = 20\%$$

That of Company B is

$$CV_B = \frac{[S_B]}{X_B} 100\% = \frac{[N4]}{N12} 100\% = 33.3\%$$

It follows that relative to the average, the share price of company B's stock is much more variable/unstable than that of Company A.

### 3.6 The Measures of Skewness

The measures of skewness are generally called Pearson's first coefficient of skewness and Pearson's second coefficient of skewness. Measures of skewness are used in determining the degree of asymmetry of a distribution; a distribution which is not symmetrical is said to be skewed.

#### 3.6.1 The Pearson's No. 1 Coefficient of Skewness

The formula used in calculating Pearson's No. 1 coefficient is:

$$Sk = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

Notice that the mean, the mode, and the standard deviation are all expressed in the units of the original data. When the difference between the mean and the mode is computed as a fraction of the standard deviation (or average spread of the data around the mean), the original units cancel out in the fraction. The result will be a coefficient of skewness, a number which tells you the extent of the skewness in the distribution.

#### Example

Consider a set of data on monthly sales of a company's product, the mean of which was found to be N240, 000; the mode found to be N135, 000; and the standard deviation found to be N85, 000. The Pearson's No. 1 Coefficient of skewness would be calculated as follows:

$$\begin{aligned} Sk &= \frac{\text{Mean} - \text{Mode}}{\sigma} = \frac{240,000 - 135,000}{85,000} \\ &= 1.24 \end{aligned}$$

Generally, a complete absence of skewness would have a coefficient of skewness equal to zero. In our example, since the mean was larger

than the mode, we obtained a positive coefficient of skewness to the extent of 124% of the standard deviation.

### 3.6.2 The Pearson's No. 2 Coefficient of Skewness

This type of the Pearson's coefficient of skewness came as a result of the fact that a precise calculation of mode is difficult in many distributions. Hence, Pearson's No. 2 coefficient of skewness uses the difference between the mean and the median of the distribution instead of the difference between the mean and the mode. In this calculation, you have the formula:

$$S_k = \frac{3(\text{mean} - \text{median})}{\sigma}$$

This formula should give you a more accurate measure of skewness than that of the Pearson's No. 1 formula.

### SELF ASSESSMENT EXERCISE 1

Why is it important for a manager to know and understand the use of the measures of dispersion or variation?

## 4.0 CONCLUSION

This unit complemented unit 6 in its presentation of the basic principles of dispersion or measures of uniformity or measures of variability in a statistical data set. Five measures of dispersion in the behaviour of business and/or economic variables were presented as: the range, the mean deviation, the variance, the standard deviation, and the coefficient of variation. Another important related measure that is primarily concerned with distribution of some business data is the Pearson Coefficient of Skewness. The computation, interpretations, and managerial implications of these measures were discussed.

## 5.0 SUMMARY

We summarise the basic measures of variations as discussed in this unit in the following form.

1. The Range (R) of a given set of ungrouped data can be determined from an ordered array as the difference between the highest observation and the lowest observation.

Let  $X_h$  = Highest observation  
 $X_L$  = Lowest observation

The  $n$ ,  $R = X_n - X_L$

2. The Mean Deviation can be defined simply by the following relationship:

$$MD = \frac{\sum |X - \bar{X}|}{n}$$

$n$  = number of observation

where  $\frac{\sum |X - \bar{X}|}{n}$  = sum of the absolute values of deviation from arithmetic mean

3. The Variance for a given set of an ungrouped data can be defined by:

$$\text{Variance} = S^2 = \frac{\sum X^2 - (\sum X/n)^2}{n - 1}$$

where  $X$  represents the numerical values of the given set of an ungrouped data.

4. The standard deviation is the most useful measure of variation. It can be defined as the square root of the variance for a given set of data.

Thus, Standard deviation =  $S = \sqrt{S^2}$

Or,

$$S = \sqrt{\frac{\sum X^2 - (\sum X)^2/n}{n-1}}, \text{ for ungrouped data}$$

5. The coefficient of variation is a relative measure. It is particularly useful when comparing the variability of two or more sets of data that are expressed in different units of measurements.

The coefficient of variation measures the standard deviation relative to the mean and is computed by:

$$\text{Coefficient of Variation} = CV = \frac{[S]}{X} 100\%$$

6. The measures of skewness are generally called Pearson's first coefficient of skewness and Pearson's second coefficient of skewness. Measures of skewness are used in determining the degree of asymmetry of a distribution; a distribution which is not symmetrical is said to be skewed.

## 6.0 TUTOR-MARKED ASSIGNMENT

The income distribution of a sample of 50 MTN subscribers in Nigeria is as follows:

Income (N000'S)	Number of subscribers
20 – 30	20
31 – 41	15
42 – 52	10
53 – 63	5

Compute:

- (a) the arithmetic mean of the subscribers' incomes
- (b) the modal and median incomes
- (c) the standard deviation of the income distribution
- (d) the coefficient of variation of the income distribution.

## 7.0 REFERENCES/FURTHER READING

Haessuler, E. F. and Paul, R. S. (1976). *Introductory Mathematical Analysis for Students of Business and Economics*, 2<sup>nd</sup> edition. Reston Virginia: Reston Publishing Company.



## **UNIT 3     STATISTICAL TOOLS III:    SETS AND SET OPERATIONS**

### **CONTENTS**

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 The Theory of Sets
    - 3.1.1 Subsets, Number of a Set, Set Equality, Universal Set, Complement of a Set, and the Venn Diagram
  - 3.2 Operations on Sets
    - 3.2.1 General Solutions to Enumeration Problems
  - 3.3 Summary of the General Enumeration Problem
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

### **1.0 INTRODUCTION**

In this unit, we develop the basic principles of probabilistic analysis, with special emphasis on the applications of set operations and events.

### **2.0 OBJECTIVES**

At the end of this unit, you should be able to:

- interpret the concept of sets as they are used in mathematical operations
- explain the theory of sets and how it is used in probability analysis.
- explain set enumerations and how they can be applied in solving business-related problems

### **3.0 MAIN CONTENT**

#### **3.1 The Theory of Sets**

The theory of sets serves as a preliminary concept necessary for the understanding of the theory of probabilities. A mathematical set is a collection of distinct objects, often referred to as *elements* or *members*.

## Examples

- (a) The employees of a company working in the Public Relations Department could be represented as:

$$PR = \{\text{Joseph, Adamu, Adebola, Nkom, Margerate}\}$$

- (b) The location of shops for a big automobile parts dealer could be represented as:

$$S = \{\text{Abuja, Enugu, Lagos, Aba, Onitsha, Kano, Ikot Ekpene}\}$$

### 3.1.1 Subsets, Number of a Set, Set Equality, Universal Set, Complement of a Set, and the Venn Diagram

#### Subsets

A subset of a set, say A, is a set which contains some of the elements of set A. For instance:

If set  $A = \{a, e, i, o, u\}$ , then:

$X = \{a, e, i\}$  is a subset of A

$Y = \{e, i\}$  is a subset of A

$Z = \{i, o, u\}$  is a subset of A

#### The Number of a Set

The *number* of a set A, written as  $n[A]$ , is defined as the number of elements in set A. For example:

If  $A = \{a, e, i, o, u\}$ , then  $n[A] = 5$  (that is, 5 elements in set A).

#### Set Equality

Two sets are said to be equal only when they have identical elements. For example:

If  $A = \{1, 2, 3\}$  and  $B = \{1, 2, 3\}$ , then Set A = Set B.

#### Universal Set

A universal set, denoted by U is a set containing different subsets of its elements. For example, a combination of different behaviours in a given population can be considered as universal, while a selected sample of such behaviours are referred to as the subsets. A set of all English

alphabets make up the universal set, while a set containing the vowels would be referred to as the subset.

### Complement of a Set

The complement of a set  $A$  (denoted by  $A'$ ) contained in a given universal set,  $U$ , is the set of elements in the universal set that are not contained in set  $A$ . For example: If set  $A$  represents the set of all skilled workers in a given universal set,  $U$ , then the complement of set  $A$ ,  $A'$ , is the set of unskilled workers who are members of the universal set.

### Venn Diagrams

Venn diagrams are simple pictorial representations of a set. They are useful for demonstrating relationships between sets.

## 3.2 Operations on Sets

There are two basic operations on sets, including:

1. set union
2. set intersection

### Set Union

The union of two sets  $X$  and  $Y$ , denoted by  $X \cup Y$ , is defined as the set which contains all the elements in  $X$  and  $Y$ . For example, if  $X = \{1, 2, 3, 4\}$  and  $Y = \{3, 4, 5\}$ , then, without repetition of elements,  $X \cup Y = \{1, 2, 3, 4, 5\}$ , so that any element of  $X$  must be an element of  $X \cup Y$ ; similarly, any element of  $Y$  must be an element of  $X \cup Y$ .

### Set Intersection

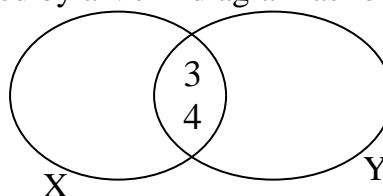
The intersection of two sets  $X$  and  $Y$  denoted by  $X \cap Y$  is defined as a set containing all the elements common to sets  $X$  and  $Y$ .

The union or intersection of three or more sets is a natural extension of the examples above and below.

As an example of intersection of two sets, consider the two sets  $X$  and  $Y$  above:

$$X \cap Y = \{3, 4\}; 3 \text{ and } 4 \text{ are common to sets } X \text{ and } Y.$$

This can be represented by a Venn diagram as follows:



The elements {3, 4} are contained in the circle common to sets X and Y.

### Set Enumeration

Set enumeration considers sets in terms of number of elements contained within the various areas defined by their union or intersection. Identifying the number of elements in these areas is known as set enumeration. As an illustration, consider the following enumeration problem.

Suppose an accounting firm currently employs 16 staff members. Given that three staff members have no formal qualifications, and of the seven staff members, who are graduates, 5 are also qualified as chartered accountants, it is possible to evaluate: (a) the number of staff who are non-graduates, chartered accountants and (b) the number of graduates who are not qualified chartered accountants.

The two values can be calculated as follows:

1. Since three staff members have no formal qualifications, there should be  $16 - 3 = 13$  staff members with at least one of the two qualifications that is, a graduate or a chartered accountant.
2. But there are 7 staff members who are graduates, which implies that  $16 - 7 - 3 = 6$  must be non-graduate, qualified chartered accountants, which gives the answer to possibility (a) above.
3. In addition, since 5 staff members are qualified chartered accountants and graduates, there would be  $7 - 5 = 2$  staff who are graduates only. This gives answer to possibility (b) above.

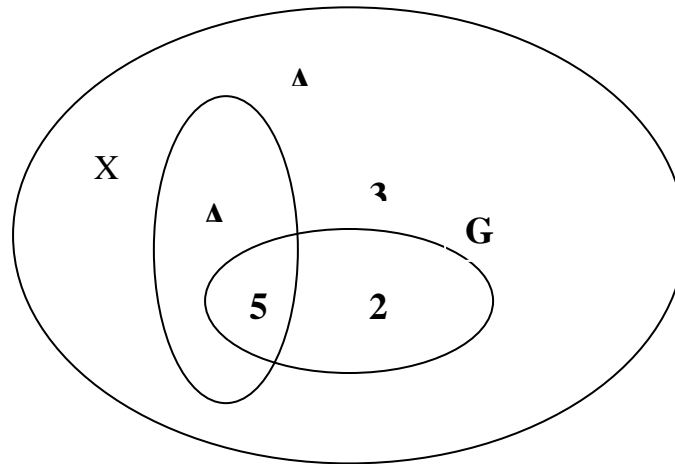
In an extended problem, the above approach is not structured enough for the solution. In the following discussions, we present a more structured procedure which solves the above problems and forms a basis for more logical approach for solving enumerations problems in general.

### 3.2.1 General Solutions to Enumeration Problems

The above problem can be solved stepwise as follows:

**Step 1:** We identify the attribute sets. The attribute sets in the problem at hand are ‘Graduate status’ (define this as set G) and ‘Qualified Chartered Accountants’ (define this as set A). Note that there is a universal set involved, that is, ‘Accounting Firm’s Staff’.

**Step 2:** We draw an **outline Venn Diagram** showing the sets that are involved:



**Figure 8.1: Universal Set of Accounting Firm’s Staff**

Do the following to form the required Venn diagram:

- (i) Draw a big circle enclosing all figures and letters above. This becomes the universal set
- (ii) Draw a circle to enclose x and 5 and label it A to form Set A
- (iii) Draw another circle to enclose 5 and 2 and label it G to form Set G.

Observe that the figure 3 stands alone in the universal set. And the figure 5 is at the intersection of Sets A and G.

The diagram contains four defined regions with values (2, 3, 5, and x), representing the number of elements in each of the regions.

**Step 3:** Use the information given to fill in as much of the Venn diagram as possible. Observe that if two sets intersect within some defined universal set, there will be four defined distinct areas. In the present example: ‘ $A \cap G$ ’; ‘A alone’; ‘G alone’; and ‘neither A nor G’, technically defined as  $(A \cup G)$ .

Since there are total of 16 staff members, the sum of the numbers in the four areas must be 16.

Interpreting the Venn diagrams above:

1. 5 staff members are graduates and qualified chartered Accountants. This number is thus entered in the ' $(A \cap G)$ ' area.
2. Since there are 7 graduates in all, there must be  $7 - 5 = 2$  graduates who are not chartered accountants. This number is thus entered in the 'G alone' area.
3. three (3) of the staff had no qualifications at all. This number is thus entered in the 'neither A nor G or  $A \cup G$ ' area.

**Step 4:** Finally, we evaluate the number of elements in any remaining unknown areas. As indicated by the above diagram,  $x$  is an unknown number in the area 'A alone'. We need to solve for  $x$  as required, that is, the number of chartered accountants who are not graduates.

Putting  $x$  as the unknown number:

$$3 + 5 + 2 + x = 16 \text{ (the total number of staff)}$$

Solving for  $x$ , we get:  $x = 6$ .

### 3.3 Summary of the General Enumeration Problems

Note in particular that there are:

- (a) 4 distinct areas for two attribute sets, and
- (b) 8 distinct areas for three attribute sets.

The general procedure for solving enumeration problems follows the following steps:

- Step 1:** Identify the attribute sets  
**Step 2:** Draw an outline Venn diagram  
**Step 3:** Use the information given to fill in as much of the diagram as possible  
**Step 4:** Evaluate the number of elements in unknown areas.

Consider the following example:

A survey was carried out by a researcher, one of the aims being to discover the extent to which computers are being used by firms in a given area. 32 firms had both stock control and payroll computerised, 65 firms had just one of these two functions computerised, and 90 firms had a computerised payroll. If 22 firms had neither of these functions computerised, how many firms were included in the survey?

**Solution**

**Step 1:** The two attributes involved are computerised payroll, with set (say P), and computerised stock control, with set (say S).

**Step 2:** Using standard notations,  
 Let p = number of firms with a computerised payroll only;  
 s = number of firms with a computerised stock control only;  
 ps = number of firms with both payroll and stock control computerised; and  
 x = number of firms with neither functions computerised,  
 construct a Venn diagram describing the situation.

**Step 3:** The following equations can be set up from the given information.

$$ps = 32 \quad (1)$$

$$p + s = 65 \quad (2)$$

$$ps + p = 90 \quad (3)$$

$$x = 22 \quad (4)$$

**Step 4:** Substituting for ps = 32 in equation 3, we get p = 58

Substituting for p = 58 in equation, we get s = 7.

It follows that the number of firms included in the survey equals:

$$p + s + ps + x = 58 + 7 + 32 + 22 = 119.$$

**SELF ASSESSMENT EXERCISE 1**

1. If  $X = \{a,4,3,d,e\}$ ;  $Y = \{3,d,e,f\}$ ; and  $Z = \{a,3,d,e,g,h\}$  within a universal set of  $U = \{a,4,3,d,e,f,g,h,i\}$ , list the elements of the following sets:

(a)  $X \cup Y$  (b)  $X \cap Y$  (c)  $Y' \cap Z'$  (d)  $X \cup Y \cap Z$

**4.0 CONCLUSION**

The theory of set is currently used in several business operations. It is used in much software programming, in the study of consumer behaviour, and in solving complex business problems. The unit has put



together some preliminary concepts in set notations. You must have been exposed to the simple operations on sets.

## 5.0 SUMMARY

The basic points in our discussions on sets and set operations can be summarised in few words as follows.

A mathematical set is a collection of distinct objects, often referred to as **elements or members**.

The **number** of a set A, written as  $n[A]$ , is defined as the number of elements in set A.

There are two basic operations on sets, including:

1. set union
2. set intersection

Set enumeration considers sets in terms of number of elements contained within the various areas defined by their union or intersection. Identifying the number of elements in these areas is known as set enumeration.

The general procedure for solving enumeration problems follows the following steps:

- Step 1:** Identify the attribute sets
- Step 2:** Draw an outline Venn diagram
- Step 3:** Use the information given to fill in as much of the diagram as possible
- Step 4:** Evaluate the number of elements in unknown areas.

## 6.0 TUTOR-MARKED ASSIGNMENT

A and B are two intersecting sets and a, b, ab, and # represent the usual symbols for the number of elements contained in the four defined areas of the associated Venn diagram. Find the values of ab and a if:  
 $n[A] = 28$ ;  $a + b = 36$ ; and,  $\# = 48$ . The number of elements in the universal set is 96

## 7.0 REFERENCES/FURTHER READING

- A. Francis (1998). *Business Mathematics and Statistics, 5<sup>th</sup> edition*. Great Britain: Ashford Colour Press.

## UNIT 4 STATISTICAL TOOLS IV: PROBABILITY THEORY AND APPLICATIONS

### CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
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  - 3.4 Joint, Marginal, Conditional Probabilities, and the Bayes Theorem
    - 3.4.1 Joint Probabilities
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  - 3.5 Probability and Expected Values
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### 1.0 INTRODUCTION

In this unit, we pay a special attention to the concept of probability, probability laws, computation of probabilities, and their applications to business decisions. At the end of this lecture, students will be expected to be able to make effective decisions under uncertainties.

The basic elements of probability theory are the outcomes of the process or phenomenon under study. Each possible type of occurrence is referred to as **an event**. The collection of all the possible events is called the **sample space**.

A **compound or joint event** is an event that has two or more characteristics. For example, the event of a student who is “an economics major and B or above average” is a joint or compound event since the student must be an economics major and have a B or above average.

The event “black ace” is also a compound event since the card must be both black and ace in order to qualify as a black ace.

**Probability** is a concept that most people understand naturally, since such words as “chance,” “likelihood,” “possibility” and “proportion are used as part of everyday speech. For example, most of the following, which might be heard in any business situation, are in fact statements of probability.

- a) “There is a 30% chance that this job will not be finished in time”.
- b) “There is every likelihood that the business will make a profit next year”.
- c) “Nine times out of ten, he arrives late for his appointments”.

In statistical sense, probability simply puts a well defined structure around the concept of everyday probability, enabling a logical approach to problem solving to be followed.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain the theory of probability
- define probability
- recognise the laws of probability
- calculate probabilities
- apply probabilities in making decisions involving uncertainties.

## 3.0 MAIN CONTENT

### 3.1 Definitions of Probability

There are basically two separate ways of calculating probability.

1. Calculation based on **theoretical probability**. This is the name given to probability that is calculated without an experiment that is, using only information that is known about the physical situation.
2. Calculation based on **empirical probability**. This is probability calculated using the results of an experiment that has been performed a number of times. Empirical probability is often referred to as *relative frequency* or *Subjective probability*.

### Definition of Theoretical Probability

Let E represent an event of an experiment that has an equally likely outcome set, U, then the theoretical probability of event E occurring when the experiment is written as Pr (E) and given by:

$$\begin{aligned} \text{Pr}(E) &= \frac{\text{number of different ways that the event can occur}}{\text{number of different possible outcomes}} \\ &= \frac{n(E)}{n(U)} \end{aligned}$$

Where n (E) = the number of outcomes in event set E

n (U) = total possible number of outcomes in outcome set, U.

If, for example, an ordinary six-sided die is to be rolled, the equally likely outcome set, U, is {1,2,3,4,5,6} and the event “even number” has event set {2,4,6}. It follows that the theoretical probability of obtaining an even number can be calculated as:

$$\text{Pr}(\text{even numbers}) = \frac{n(\text{even numbers})}{n(U)} = \frac{3}{6} = 0.50$$

### Other Examples

A wholesaler stocks heavy (2B), medium (HB), fine (2H) and extra fine (3H) pencils which come in packs of 10. Currently in stock are 2 packs of 3H, 14 packs of 2H, 35 packs of HB and 8 packs of 2B. If a pack of pencil is chosen at random for inspection, what is the probability that they are:

- (a) medium      (b) heavy      (c) not very fine  
 (d) neither heavy nor medium?

### Solutions

Since the pencil pack is chosen at random, each separate pack of pencils can be regarded as a single equally likely outcome. The total number of outcomes is the number of pencil packs, that is,  $2+14+35+8 = 59$ .

Thus,  $n(U) = 59$

$$(a) \quad \Pr(\text{medium}) = \frac{n(\text{medium})}{n(U)} = \frac{35}{59} = 0.593$$

$$(b) \quad \Pr(\text{heavy}) = \frac{n(\text{heavy})}{n(U)} = \frac{8}{59} = 0.136$$

(c)  $\Pr(\text{not very fine})$ . Note that the number of pencil packs that are not very fine is  $14+35+8 = 57$

$$\text{Therefore, } \Pr(\text{not very fine}) = \frac{n(\text{not very fine})}{n(U)} = \frac{57}{59} = 0.966$$

(d) “Neither heavy nor medium” is equivalent to “fine” or “very fine” in the problem. There is  $2+14 = 16$  of these pencil packs.

Thus,  $\Pr(\text{neither heavy nor medium})$

$$= \frac{n(\text{neither heavy nor medium})}{n(U)} = \frac{16}{59} = 0.271$$

### Definition of Empirical (Relative Frequency) Probability

If E is some event of an experiment that has been performed a number of times, yielding a frequency distribution of events or outcomes, then the empirical probability of event E occurring when the experiment is performed one more time is given by:

$$\Pr(E) = \frac{\text{number of times the event occurred}}{\text{number of times the experiment was performed}} = \frac{f(E)}{\sum f}$$

Where  $f(E)$  = the frequency of event E

$\sum f$  = total frequency of the experiment.

Put differently, the empirical probability of an event E occurring is simply the proportion of times that event E actually occurred when the experiment was performed.

For example, if, out of 60 orders received so far this financial year, 12 were not completely satisfied, the proportion,  $12/60 = 0.2$  is the empirical probability that the next order received will not be completely satisfied.

### Other Examples

A number of families of a particular type were measured by the number of children they have, given the following frequency distribution:

Number of children:	0	1	2	3	4	5 or more
Number of families:	12	28	22	8	2	2

Use this information to calculate the (relative frequency) probability that another family of this type chosen at random will have:

- (a) 2 children      (b) 3 or more children  
 (c) less than 2 children

### Solutions

Here,  $\Sigma f$  = total number of families = 74

$$(a) \quad \Pr(2 \text{ children}) = \frac{f(2 \text{ children})}{\Sigma f} = \frac{22}{74} = 0.297$$

$$(b) \quad f(3 \text{ or more children}) = 8+2+2 = 12$$

$$\text{Thus, } \Pr(3 \text{ or more children}) = \frac{12}{74} = 0.162$$

$$(c) \quad \Pr(\text{less than 2 children}) = \frac{f(\text{less than 2 children})}{\Sigma f} = \frac{12+28}{74} = 0.541$$

## 3.2 Laws of Probability

There are four basic laws of probability.

1. Addition Law for mutually exclusive events
2. Addition Law for events that are not mutually exclusive
3. Multiplication Law for Independent events
4. Multiplication Law for Dependent events.

### **Addition Law for Mutually Exclusive Events**

Two events are said to be mutually exclusive events if they cannot occur at the same time. The addition law states that if events A and B are mutually exclusive events, then:  $\Pr (A \text{ or } B) = \Pr (A) + \Pr (B)$

### **Examples**

The purchasing department of a big company has analysed the number of orders placed by each of the 5 departments in the company by type as follows:

**Table 9.1: Departmental Orders**

Type of order	Department					
	Sales	Purchasing	Production	Accounts	Maintenance	Total
Consumables	10	12	4	8	4	38
Equipment	1	3	9	1	1	15
Special	0	0	4	1	2	7
Total	11	15	17	10	7	60

An error has been found in one of these orders. What is the probability that the incorrect order:

- came from maintenance?
- came from production?
- came from maintenance or production?
- came from neither maintenance nor production?

### Solutions

- a) Since there are 7 maintenance orders out of the 60,

$$\Pr(\text{maintenance}) = \frac{7}{60} = 0.117$$

- b) Similarly,  $\Pr(\text{Production}) = \frac{17}{60} = 0.283$

- c) Maintenance and production departments are two mutually exclusive events so that,

$$\Pr(\text{maintenance or production}) = \Pr(\text{maintenance}) + \Pr(\text{production}) = 0.117 + 0.283 = 0.40$$

- d)  $\Pr(\text{neither maintenance nor production})$   
 $= 1 - \Pr(\text{maintenance or production}) = 1 - 0.4 = 0.6$

### Addition Law for Events that are Not Mutually Exclusive Events

If events A and B are not mutually exclusive, that is, they can either occur together or occur separately, then according to the Law:

$$\Pr(A \text{ or } B \text{ or Both}) = \Pr(A) + \Pr(B) - \Pr(A) \cdot \Pr(B)$$

### Example

Consider the following contingency table for the salary range of 94 employees:



**Table 9.2: Contingency Table for the Salary range of 94 Employees**

Salary/month	Men	Women	Total
₱10,000 and above	20	37	57
Below 10,000	15	22	37
Total	35	59	94

What is the probability of selecting an employee who is a man or earns below ₱10, 000 per month?

**Solution**

The two events of being a man and earning below ₱10, 000 is not mutually exclusive.

It follows that:

$$\Pr(\text{man or below N10,00}) = \Pr(\text{man}) + \Pr(\text{below N10,000}) - \Pr(\text{man}) \cdot \Pr(\text{below N10,000})$$

$$\begin{aligned} &= \frac{35}{94} + \frac{37}{94} - \frac{(35)(37)}{(94)(94)} \\ &= 0.372 + 0.394 - (0.372)(0.394) \\ &= 0.766 - 0.147 \\ &= 0.619 \text{ or } 61.9\% \end{aligned}$$

**Multiplication Law for Independent Events**

This law states that if A and B are independent events, then:

$$\Pr(A \text{ and } B) = \Pr(A) \cdot \Pr(B)$$

As an **example**, suppose, in any given week, the probability of an assembly line failing is 0.03 and the probability of a raw material shortage is 0.1.

If these two events are independent of each other, then the probability of an assembly line failing and a raw material shortage is given by:

$$\begin{aligned} \Pr(\text{Assembly line failing and Material shortage}) &= (0.03)(0.1) \\ &= 0.003 \end{aligned}$$

**Multiplication Law for Dependent Events**

This Law states that if A and B are dependent events, then:

$$\Pr(A \text{ and } B) = \Pr(A) \cdot \Pr(B/A)$$

Note that  $\Pr(B/A)$  is interpreted as probability of B given that event A has occurred.

### Example

A display of 15 T-shirts in a Sports shop contains three different sizes: small, medium and large. Of the 15 T-shirts:

3 are small

6 are medium

6 are large.

If two T-shirts are randomly selected from the T-shirts, what is the probability of selecting both a small T-shirt and a large T-shirt, the first not being replaced before the second is selected?

### Solution

Since the first selected T-shirt is not replaced before the second T-shirt is selected, the two events are said to be dependent events.

It follows that:

$$\begin{aligned} \Pr(\text{Small T-shirt and Large T-shirt}) &= \Pr(\text{Small}) \cdot \Pr(\text{Large/Small}) \\ &= (3/15)(4/14) \\ &= (0.2)(0.429) \\ &= 0.086 \end{aligned}$$

### 3.3 Computational Formula for Multiple Occurrence of an Event

The probability of an event, E occurring X times in n number of trials are given by the formula:

$$\Pr(E_x^n) = C_{n,x} p^x q^{(n-x)}$$

$$\text{Where } C_{n,x} = \frac{n!}{X!(n-x)!}$$

p = probability of success

q = probability of failure

$$p + q = 1$$

**Example**

Assume there is a drug store with 10 antibiotic capsules of which 6 capsules are effective and 4 are defective. What is the probability of purchasing the effective capsules from the drug store?

**Solution**

From the given information: The probability of purchasing an effective capsule is:

$$P = 6/10 = 0.60$$

Since  $p + q = 1$ ;  $q = 1 - 0.60 = 0.40$ ;  $n = 10$ ;  $x = 6$

$$\begin{aligned} \Pr (E^10_6) &= \text{probability of purchasing the 6 effective capsules} \\ &= C_{10,6}(0.6)^6(0.4)^4 \\ &= 10!/(6!(10 - 6!)(0.047)(0.026) \\ &= 10.9.8.7.6!(0.0012) \\ &\quad \quad \quad 6! 4! \\ &= 210(0.0012) = 0.252 \end{aligned}$$

Hence, the probability of purchasing the 6 effective capsules out of the 10 capsules is 25.2 percent

### **3.4 Joint, Marginal, Conditional Probabilities, and the Bayes Theorem**

#### **3.4.1 Joint Probabilities**

A joint probability implies the probability of joint events. Joint probabilities can be conveniently analysed with the aid of joint probability tables.

#### **The Joint Probability Table**

A joint probability table is a contingency table in which all possible events for a variable are recorded in a row and those of other variables

are recorded in a column, with the values listed in corresponding cells as in the following example.

### Example

Consider a research activity with the following observations on the number of customers that visit XYZ supermarket per day. The observations (or events) are recorded in a joint probability table as follows:

**Table 9.3: Joint Probability Table**

Age (Years)	Male (M)	Female (F)	Total
Below 30 (B)	60	70	130
30 and Above (A)	60	20	80
Total	120	90	210

We can observe four joint events from the above table:

Below 30 and Male ( $B \cap M$ ) = 60

Below 30 and Female ( $B \cap F$ ) = 70

30 and Above and Male ( $A \cap M$ ) = 60

30 and Above and Female ( $A \cap F$ ) = 20

Total events or sample space = 210

The **joint probabilities** associated with the above joint events are:

$$\Pr(B \cap M) = \frac{60}{210} = 0.2857$$

$$\Pr(B \cap F) = \frac{70}{210} = 0.3333$$

$$\Pr(A \cap M) = \frac{60}{210} = 0.2857$$

$$\Pr(A \cap F) = \frac{20}{210} = 0.0952$$

### 3.4.2 Marginal Probabilities

The Marginal Probability of an event is its simple probability of occurrence, given the sample space. In the present discussion, the results of adding the joint probabilities in rows and columns are known as marginal probabilities.

The marginal probability of each of the above events:

Male (M), Female (F), Below 30 (B), and  
Above 30 (A) are as follows:

$$\Pr(M) = \Pr(B \cap M) + \Pr(A \cap M) = 0.2857 + 0.2857 = 0.57$$

$$\Pr(F) = \Pr(B \cap F) + \Pr(A \cap F) = 0.3333 + 0.0952 = 0.43$$

$$\Pr(B) = \Pr(B \cap M) + \Pr(B \cap F) = 0.2857 + 0.3333 = 0.62$$

$$\Pr(A) = \Pr(A \cap M) + \Pr(A \cap F) = 0.2857 + 0.0952 = 0.38$$

The joint and marginal probabilities above can be summarised in a contingency table as follows:

**Table 9.4: Joint and Marginal Probability Table.**

Age	Male (M)	Female (F)	Marginal Probability
Below 30 (B)	0.2857	0.3333	0.62
30 and Above (A)	0.2857	0.0952	0.38
Marginal Probability	0.57	0.43	1.00

### 3.4.3 Conditional Probability

Assuming two events, A and B, the probability of event A, given that event B has occurred is referred to as the conditional probability of event A.

In symbolic term:

$$\Pr(A/B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A) \cdot \Pr(B)}{\Pr(B)} = \Pr(A)$$

Where  $\Pr(A/B)$  = conditional probability of event A  
 $\Pr(A \cap B)$  = joint probability of events A and B  
 $\Pr(B)$  = marginal probability of event B

In general,

$$\Pr(A/B) = \frac{\text{Joint Probability of events A and B}}{\text{Marginal Probability of event B}}$$

### 3.4.4 The Bayes Theorem

Bayes theorem is a formula which can be thought of as “reversing” conditional probability. That is, it finds a conditional probability, A/B given, among other things, its inverse, B/A.

According to the theorem, given events A and B,

$$\Pr(A/B) = \frac{\Pr(A) \cdot \Pr(B/A)}{\Pr(B)}$$

As an example in the use of Bayes theorem, if the probability of meeting a business contract date is 0.8, the probability of good weather is 0.5 and the probability of meeting the date given good weather is 0.9, we can calculate the probability that there was good weather given that the contract date was met.

Let G = good weather, and m = contract date was met

Given that:  $\Pr(m) = 0.8$ ;  $\Pr(G) = 0.5$ ;  $\Pr(m/G) = 0.9$ , we need to find  $\Pr(G/m)$ :

From the Bayes theorem:

$$\begin{aligned} \Pr(G/m) &= \frac{\Pr(G) \cdot \Pr(m/G)}{\Pr(m)} = \frac{(0.5)(0.9)}{0.8} \\ &= 0.5625 \text{ or } 56.25\% \end{aligned}$$

### 3.5 Probability and Expected Values

The expected value of a set of values, with associated probabilities, is the arithmetic mean of the set of values. If some variable, X, has its values specified with associated probabilities, P, then:

$$\text{Expected value of X} = E(X) = \sum PX$$

#### Example

An ice-cream salesman divides his days into ‘Sunny’ ‘Medium’ or ‘Cold’. He estimates that the probability of a sunny day is 0.2 and that 30% of his days are cold. He has also calculated that his average revenue on the three types of days is N220, N130, and N40 respectively. If his average total cost per day is N80, calculate his expected profit per day.

### Solution

We first calculate the different values of profit that are possible since we are required to calculate expected profit per day, as well as their respective probabilities.

Given that  $\Pr(\text{sunny day}) = 0.2$ ;  $\Pr(\text{cold day}) = 0.3$

Since in theory,  $\Pr(\text{sunny day}) + \Pr(\text{cold day}) + \Pr(\text{medium day}) = 1$

It follows that:

$$\Pr(\text{medium day}) = 1 - 0.2 - 0.3 = 0.5$$

The total costs are the same for any day (N80), so that the profits that the salesman makes on each day of the three types of day are:

$$\begin{aligned} \text{Sunny day: } & N(220-80) = N140 \\ \text{Medium day: } & N(130-80) = N50 \\ \text{Cold day: } & N(40-80) = -N40 \text{ (loss)} \end{aligned}$$

We can summarise the probability table as follows:

**Table 9.5: Probability Table**

	Sunny	Medium	Cold
Profit (N)	140	50	-40
Probability	0.2	0.5	0.3

From this Table, the expected profit per day is:

$$E(\text{profit}) = N(0.2(140) + 0.5(50) - 0.3(40)) = N41$$

### SELF ASSESSMENT EXERCISE 2

1. What do we mean by the terms “mutually exclusive events” and “independent events?”
2. State, with simple examples, the four laws of probability.

## 4.0 CONCLUSION

**Probability** is a concept that most people understand naturally, since such words as “chance,” “likelihood,” “possibility” and “proportion are

used as part of everyday speech. It is a term used in making decisions involving uncertainty. Though the concept is often viewed as very abstract and difficult to relate to real world activities, it remains the best tool for solving uncertainties problems.

To remove some of the abstract nature of probabilities, this unit has provided you with the simplest approach to understanding and calculating, as well as applying the probability concept. It defines probability in two basic forms:

- (i) the theoretical definition; and
- (ii) the empirical definition.

## 5.0 SUMMARY

The issues discussed in this unit can be summarised in the following way:

there are basically two separate ways of calculating probability which are as stated below:

- i. **theoretical probability**: this is calculated without an experiment, that is, using only information that is known about the physical situation.
- ii. calculation based on **empirical probability**. This is probability calculated using the results of an experiment that has been performed a number of times. Empirical probability is often referred to as **Relative frequency** or **Subjective probability**.

There are four basic **laws of probability**:

1. addition law for mutually exclusive events
2. addition law for events that are not mutually exclusive
3. multiplication law for independent events
4. multiplication law for dependent events.

The probability of an event, E occurring an X time in n number of trials is given by the formula:

$$\Pr (E^n_x) = C_{n,x} p^x q^{(n-x)}$$

$$\text{Where } C_{n,x} = \frac{n!}{X!(n-x)!}$$

p = probability of success



$$q = \text{probability of failure}$$
$$p + q = 1$$

A **joint probability** implies the probability of joint events. Joint probabilities can be conveniently analysed with the aid of joint in which all possible events for a variable are recorded in a row and those of other variables are recorded in a column, with the values listed in corresponding cells.

The **Marginal Probability** of an event is its simple probability of occurrence, given the sample space.

Assuming two events, A and B, the probability of event A, given that event B has occurred is referred to as the **conditional probability** of event A.

The expected value of a set of values, with associated probabilities, is the arithmetic mean of the set of values. If some variable, X, has its values specified with associated probabilities, P, then:

$$\text{Expected value of } X = E(X) = \sum PX$$

## 6.0 TUTOR-MARKED ASSIGNMENT

A firm has tendered for two independent contracts. It estimates that it has probability 0.4 of obtaining contract A and probability 0.1 of obtaining contract B. Find the probability that the firm:

- (a) obtains both contracts
- (b) obtains neither of the contracts
- (c) obtains exactly one contract

## 7.0 REFERENCES/FURTHER READING

- A. Francis (1998). *Business Mathematics and Statistics, 5<sup>th</sup> edition*. Great Britain: Ashford Colour Press.

## UNIT 5 CORRELATION THEORY

### CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Linear Correlation
  - 3.2 *Rank Correlation*
  - 3.3 Partial Correlations
  - 3.4 *Limitations of the Theory of Linear Correlations*
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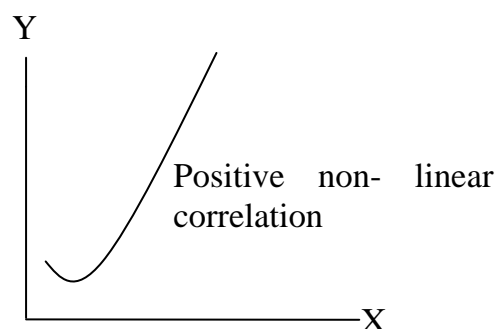
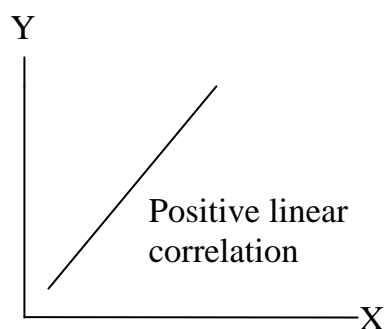
### 1.0 INTRODUCTION

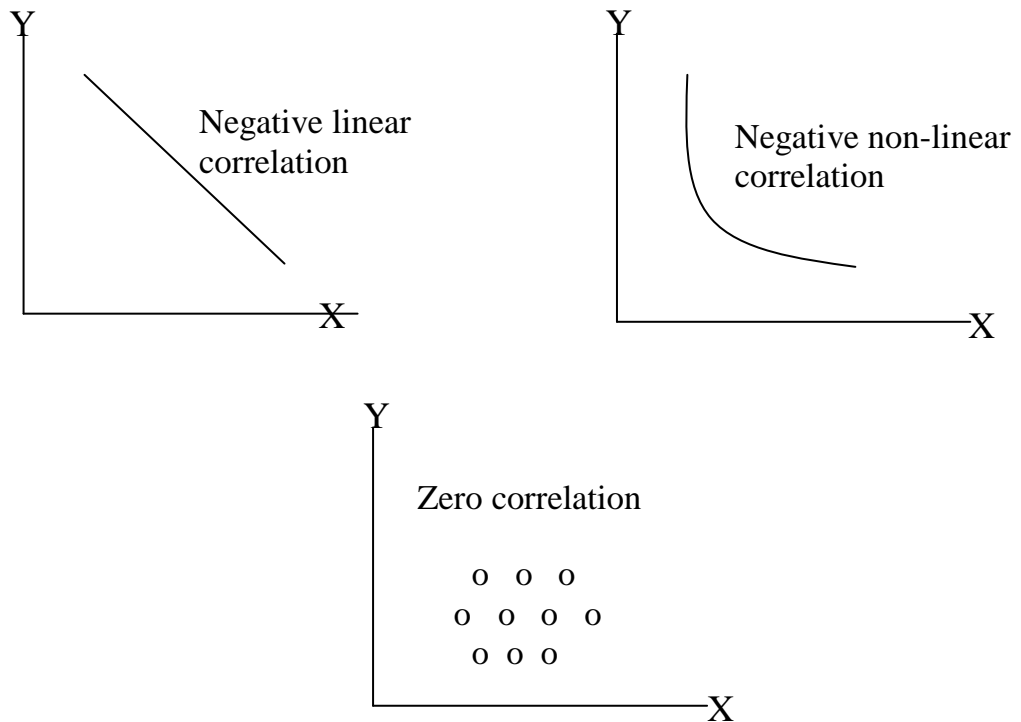
The simplest methods of measuring relationships existing between economic variables are correlation analysis and regression analysis.

Correlation can be defined as the degree of relationship between two or more variables. The degree of relationship between two variables is called simple correlation. The degree of relationship existing among three or more variables is called multiple correlations.

Correlation may be linear for scatter diagram on the values of two variables, (X and Y) are clustered near a straight line, or nonlinear, when all points on the scatter lie near a curve.

Two variables may have a positive correlation or a negative correlation, or they may be unrelated. These correlations are represented in the following diagrams:





## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- describe the computation of linear correlation coefficients
- explain the computation of rank correlation coefficients
- explain the computation of partial correlation coefficients
- apply the concept of correlations in business decisions.

## 3.0 MAIN CONTENT

### 3.1 Linear Correlation

We can determine the kind of correlation between two variables by direct observation of the scatter diagrams. In addition, the scatter diagram indicates the strength of the relationship between the two variables. If the points lie close to the line, the correlation is strong. A greater dispersion of points about the line implies weaker correlation.

Note that the inspection of a scatter diagram only gives a rough picture of the relationship between two variables. For a precise quantitative measurement of the degree of correlation between two variables, say X and Y, we use a parameter  $\rho$  referred to as the correlation coefficient. The sample estimate of this parameter is referred to as r.

The simple correlation coefficient is computed by:

$$r_{xy} = \frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{n\Sigma(X_i - \bar{X})^2} \sqrt{\Sigma(Y_i - \bar{Y})^2}}$$

$$= \frac{n\Sigma X_i Y_i - \Sigma(X_i)(\Sigma Y_i)}{\sqrt{n\Sigma X_i^2 - (\Sigma X_i)^2} \sqrt{n\Sigma Y_i^2 - (\Sigma Y_i)^2}}$$

### Example

We want to determine the correlation between Price, X, and quantity supplied, Y, Given the following data

Time Period (in days)	Quantity supply (Y <sub>i</sub> ) (in tons)	Unit Price (X <sub>i</sub> ) (in Naira)
1	10	2
2	20	4
3	50	6
4	40	8
5	50	10
6	60	12
7	80	14
8	90	16
9	90	18
10	120	20
n = 10	$\Sigma Y_i = 610$	$\Sigma X_i = 110$

$$\bar{X} = \frac{\Sigma X_i}{n} = \frac{110}{10} = 11, \bar{Y} = \frac{\Sigma Y_i}{n} = \frac{610}{10} = 61$$

It can be shown that the formula:

$$r = \frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{n\Sigma(X_i - \bar{X})^2} \sqrt{\Sigma(Y_i - \bar{Y})^2}}$$

$$= \frac{n\sum X_i Y_i - (\sum X_i)(\sum Y_i)}{\sqrt{n\sum X_i^2 - (\sum X_i)^2} \sqrt{n\sum Y_i^2 - (\sum Y_i)^2}}$$

Using the table

$X_i$	$Y_i$	$X_i Y_i$	$X_i^2$	$Y_i^2$
2	10	20	4	100
4	20	80	16	400
6	50	300	36	2,500
8	40	320	64	1,600
10	50	500	100	2,500
12	60	720	144	3,600
14	80	1120	196	6,400
16	90	1,440	256	8,100
18	90	1620	324	8100
20	120	2400	400	14400

$$\begin{aligned}
 r &= \frac{n \sum X_i Y_i - (\sum X_i)(\sum Y_i)}{\sqrt{n \sum X_i^2 - (\sum X_i)^2} \sqrt{n \sum Y_i^2 - (\sum Y_i)^2}} \\
 &= \frac{10(8520) - (110)(610)}{\sqrt{10(1540) - (110)^2} \sqrt{10(47700) - (610)^2}} \\
 &= \frac{85200 - 67100}{\sqrt{15400 - 12100} \sqrt{477000 - 372100}} \\
 &= \frac{18100}{\sqrt{3300} \sqrt{104900}} = \frac{18100}{(57.45)(323.88)} \\
 &= \frac{18100}{18606.91} = 0.972 \text{ approx.}
 \end{aligned}$$

### 3.3 Rank Correlation

Rank correlation coefficient is used for qualitative variables, whereby the variables cannot be measured numerically. Examples of such variables include profession, education, preferences for a particular brand of commodity and the like.

In such cases, we rank the observations in a specific sequence, such as in order of size, importance, etc., using numbers 1, 2... n. We therefore, assign ranks to the data and measure the relationships between their ranks instead of their actual numerical values.

If two variables, X and Y are ranked, the rank correlation coefficient is computed by:

$$r_s = 1 - \frac{6 \sum d^2}{n^3 - n}$$

Where  $d^2 = [R(X)-R(Y)]^2 =$  squared difference between ranks of corresponding pairs of X and Y.

R(X) = individual ranks assigned to X values

R(Y) = individual ranks assigned to Y values

n = number of pairs of observations

There are two important points to note about the rank correlation coefficient:

1. It does not matter whether the ranking of the observations is in ascending or descending order.
2. If two or more observations have the same value, we assign to each observation the **Mean rank**.

### Examples

1. Consider the following table showing how 10 students were ranked according to their performance in their class work and their final examinations. Let us find out whether there is a relationship between the accomplishments of the students during the year and their performance in their examinations.

Student	Ranking Based on Class Work	Ranking Based on Exam. Marks	d	d <sup>2</sup>
A	2	1	1	1
B	5	6	-1	1
C	6	4	2	4
D	1	2	-1	1
E	4	3	1	1
F	10	7	3	9
G	7	8	-1	1
H	9	10	-1	1
I	3	5	-2	4
J	8	9	-1	1

$$\sum d^2 = 24$$

The correlation coefficient in this example is:

$$r_s = 1 - \frac{6 \sum d^2}{n^3 - n} = 1 - \frac{6(24)}{10^3 - 10} = 1 - 0.145 = 0.855$$



The value of  $r_s = 0.855$  indicates a close relationship between class work and examination performance, so that students with good records throughout the year do well in their examination and vice versa.

- In a recent research project, a student attempted to find out if the demand for one specific product is affected by expenditures on television advertisements.

The following data were obtained from the reference firm:

Year	Qd (in thousands) (Q)	Expenditure on TV Adverts (X) (N <sup>2</sup> millions)	R(X)	R(Q)	d	d <sup>2</sup>
					R(x)-R(Q)	{R(x) – R(Q)} <sup>2</sup>
1980	35.00	0.05	3.0	3.5	-0.50	0.025
1981	37.50	0.09	7.0	5.0	2.00	4.00
1982	39.20	0.04	1.5	7.0	-5.50	30.25
1983	22.00	0.04	1.5	1.0	0.50	0.25
1984	33.00	0.10	8.5	2.0	6.50	42.25
1985	41.60	0.10	8.5	8.0	0.50	0.25
1986	44.80	0.11	10.	9.0	1.00	1.00
1987	35.00	0.08	6.0	3.5	2.50	6.25
1988	38.00	0.06	4.0	6.0	-2.00	4.00
1989	51.80	0.12	11.0	10.0	1.00	1.00
1990	62.05	0.14	13.0	12.0	1.00	1.00
1991	63.00	0.13	12.0	13.0	-1.00	1.00
1992	58.30	0.07	5.0	11.0	-6.00	36.00
1993	70.00	0.20	14.0	15.0	-1.00	1.00
1994	68.08	0.21	15.0	14.0	-1.00	1.00
1995	72.00	0.27	16.0	16.0	0.00	0.00
						$\Sigma d^2 = 129.50$

The rank correlation coefficient can be computed as follows:

$$\begin{aligned}
 r_s &= 1 - \frac{6 \sum d^2}{n^3 - n} = 1 - \frac{6(129.50)}{16^3 - 16} \\
 &= 1 - \frac{777}{4080} = 0.81
 \end{aligned}$$

The value,  $r_s = 0.81$  indicates a strong positive correlation between the quantity demanded of the specific product (Q) and expenditure on TV adverts (X).

### 3.3 Partial Correlations

A *partial correlation coefficient* measures the relationship between any two variables, keeping other variables constant.

Assume a multiple relationship between three variables,  $X_1$ ,  $X_2$ , and  $X_3$ . To measure the true correlation between  $X_1$  and  $X_2$ , we find the partial correlation coefficient between  $X_1$  and  $X_2$ , keeping  $X_3$  constant.

The partial correlation coefficient is determined in terms of the simple correlation coefficients among the various variables involved in a multiple relationship. For the three variables,  $X_1$ ,  $X_2$  and  $X_3$ , three simple correlation coefficients are involved as follows:

- (1)  $r_{12}$  = correlation coefficient between  $X_1$  and  $X_2$
- (2)  $r_{13}$  = correlation coefficient between  $X_1$  and  $X_3$
- (3)  $r_{23}$  = correlation coefficient between  $X_2$  and  $X_3$ .

There are also two partial correlation coefficients involved:

- (1)  $r_{12.3}$  = partial correlation coefficient between  $X_1$  and  $X_2$ , keeping  $X_3$  constant.

This is given by:

$$r_{12.3} = \frac{r_{12} - (r_{13})(r_{23})}{\sqrt{[1 - (r_{13})^2][1 - (r_{23})^2]}}$$

- (2)  $r_{13.2}$  = partial correlation coefficient between  $X_1$  and  $X_3$ , keeping  $X_2$  constant:

$$r_{13.2} = \frac{r_{13} - (r_{12})(r_{23})}{\sqrt{[1 - (r_{12})^2][1 - (r_{23})^2]}}$$

The formula for partial correlation coefficient can be extended to relationships involving any number of explanatory variables.

#### Examples

An automobile dealer observes the following results on his business variables for 6 months:

	Month					
Variable	1	2	3	4	5	6

$X_1$	10	10	20	30	40	40
$X_2$	0	1	2	2	3	4
$X_3$	1	0	2	3	3	3

where,

$X_1$  = number of new cars sold per month

$X_2$  = number of 10 minute local TV spot advert during the month

$X_3$  = number of full-page newspaper advert during the month.

- (a) Compute the simple correlation coefficient:
- between  $X_1$  and  $X_2$ , ( $r_{12}$ )
  - between  $X_1$  and  $X_3$ , ( $r_{13}$ )
  - between  $X_2$  and  $X_3$ , ( $r_{23}$ )
- (b) Compute the partial correlation coefficient:
- between  $X_1$  and  $X_2$ , keeping  $X_3$  constant, ( $r_{12.3}$ )
  - between  $X_1$  and  $X_3$ , keeping constant, ( $r_{13.2}$ )

### Solutions

$$a. \quad (i) \quad r_{12} = \frac{n\sum X_1X_2 - \sum X_1\sum X_2}{\sqrt{n\sum X_1^2 - (\sum X_1)^2}\sqrt{n\sum X_2^2 - (\sum X_2)^2}}$$

	$X_1$	$X_2$	$X_1^2$	$X_2^2$	$X_1X_2$
	10	0	100	0	0
	10	1	100	1	10
	20	2	400	4	40
	30	2	900	4	60
	40	3	1600	9	120
	40	4	1600	16	160
Total	150	12	4700	34	390

$$n\sum X_1X_2 = 6(390) = 2340$$

$$\sum X_1\sum X_2 = 150(12) = 1800$$

$$n\sum X_1^2 = 6(4700) = 28,200$$

$$n\sum X_2^2 = 6(34) = 204$$

$$(\sum X_1)^2 = (150)^2 = 22500$$

$$(\sum X_2)^2 = (12)^2 = 144$$

$$\begin{aligned}
 r_{12} &= \frac{2340 - 1800}{\sqrt{28200 - 22500}\sqrt{204 - 144}} \\
 &= \frac{540}{(75.4983)(7.7460)} \\
 &= \frac{540}{584.81} = 0.923
 \end{aligned}$$

This implies a very close relationship between the number of cars sold per month and the number of 10-minute local TV spot adverts during the month.

$$\text{a. (ii) } r_{13} = \frac{n\sum X_1X_3 - \sum X_1\sum X_3}{\sqrt{n\sum X_1^2 - (\sum X_1)^2}\sqrt{n\sum X_3^2 - (\sum X_3)^2}}$$

Using the table below:

	$X_1$	$X_3$	$X_1^2$	$X_3^2$	$X_1X_3$
	10	1	100	1	10
	10	0	100	0	0
	20	2	400	4	40
	30	3	900	9	90
	40	3	1600	9	120
	40	3	1600	9	120
<b>TOTAL</b>	<b>150</b>	<b>12</b>	<b>4700</b>	<b>32</b>	<b>380</b>

$$n\sum X_1X_3 = 6(380) = 2280$$

$$\sum X_1\sum X_3 = 150(12) = 1800$$

$$n\sum X_1^2 = 6(4700) = 28200$$

$$n\sum X_3^2 = 6(32) = 192$$

$$(\sum X_1)^2 = (150)^2 = 22500$$

$$(\sum X_3)^2 = (12)^2 = 144$$

$$\begin{aligned}
 r_{13} &= \frac{2280 - 1800}{\sqrt{28200 - 22500}\sqrt{192 - 144}} \\
 &= \frac{480}{(75.4983)(6.9282)} \\
 &= 480/523.0673 = 0.918
 \end{aligned}$$

Again, the coefficient implies a strong relationship between number of new cars sold per month and number of full page newspaper advert during the month.

$$a. \quad (iii) \quad r_{23} = \frac{n \sum X_2 X_3 - \sum X_2 \sum X_3}{\sqrt{n \sum X_2^2 - (\sum X_2)^2} \sqrt{n \sum X_3^2 - (\sum X_3)^2}}$$

The expressions in the formula can be calculated using the following table:

$X_2$	$X_3$	$X_2^2$	$X_3^2$	$X_2 X_3$
0	1	0	1	0
1	0	1	0	0
2	2	4	4	4
2	3	4	9	6
3	3	9	9	9
4	3	16	9	12
<b>Total = 12</b>	<b>12</b>	<b>34</b>	<b>32</b>	<b>31</b>

$$n \sum X_2 X_3 = 6(31) = 186$$

$$\sum X_2 \sum X_3 = 12(12) = 144$$

$$n \sum X_2^2 = 6(34) = 204$$

$$(\sum X_2)^2 = (12)^2 = 144$$

$$n \sum X_3^2 = 6(32) = 192$$

$$(\sum X_3)^2 = (12)^2 = 144$$

$$\begin{aligned} r_{23} &= \frac{186 - 144}{\sqrt{(204 - 144)} \sqrt{(192 - 144)}} \\ &= 42 / (7.746)(6.9282) \\ &= 42/53 = 0.783 \end{aligned}$$

b (i) Using the figures obtained in a(i), a(ii), and a(iii) above, we compute the partial correlation coefficients,  $r_{12.3}$  as follows:

$$\begin{aligned} r_{12.3} &= \frac{r_{12} - (r_{13})(r_{23})}{\sqrt{[1 - (r_{13})^2][1 - (r_{23})^2]}} \\ &= \frac{0.923 - (0.918)(0.783)}{\sqrt{[1 - (0.918)^2][1 - (0.783)^2]}} \end{aligned}$$

$$\begin{aligned}
&= \frac{0.923 - 0.7188}{\sqrt{(0.1573)(0.3869)}} \\
&= \frac{0.2042}{\sqrt{0.6082}} \\
&= 0.2042/0.2467 = 0.8277
\end{aligned}$$

It follows that the partial correlation between the number of new cars sold per month,  $X_1$ , and the number of 10-minute local TV spot advert during the month,  $X_2$ , keeping the number of full-page advert,  $X_3$ , constant is high.

b (ii) The partial correlation between the number of new cars sold per month and the number of full-page newspaper advert during the month, keeping the number of 10-minute advert during the month,  $r_{13.2}$ , can be similarly obtained as follows:

$$\begin{aligned}
r_{13.2} &= \frac{r_{13} - (r_{12})(r_{23})}{\sqrt{[1 - (r_{12})^2][1 - (r_{23})^2]}} \\
&= \frac{0.918 - (0.923)(0.783)}{\sqrt{[1 - (0.923)^2][1 - (0.783)^2]}} \\
&= \frac{0.1953}{\sqrt{(0.1489)(0.3869)}} \\
&= 0.1953/0.2400 = 0.8138
\end{aligned}$$

The computed coefficient indicates a high partial correlation between the number of new cars sold and the number of full-page advert during the month, keeping the number of 10-minute spot advert constant.

### 3.4 Limitations of the Theory of Linear Correlations

The limitations of linear correlations as a technique for the study of economic relations are as follows:

1. The formula for correlation coefficient applies only to linear relationships between variables.
2. That correlation coefficient is a measure of co-variability of variables does not imply any functional relationship between the variables concerned.

**SELF ASSESSMENT EXERCISE 1**

1. How does simple correlation differ from partial correlation?
2. Use graphs to illustrate: positive linear correlation, negative nonlinear correlation, and zero correlation between two variables

## 4.0 CONCLUSION

The relationships among business variables can simply be identified using correlation coefficients. Two variables can either be positively or negatively correlated. This correlation can be linear or nonlinear depending on variable characteristics.

Three basic types of correlations were discussed in this unit: (i) the simple linear correlation; (ii) rank correlation; and, (iii) partial correlation. You should note that partial correlations involve more than two different variables. Here, the correlation between two identified variables, are obtained holding other variables constant.

## 5.0 SUMMARY

For a precise quantitative measurement of the degree of correlation between two variables, say X and Y, we use a parameter  $\rho$  referred to as the correlation coefficient. The sample estimate of this parameter is referred to as r.

**Rank Correlation Coefficient** is used for qualitative variables, whereby the variables cannot be measured numerically. Examples of such variables include profession, education, preferences for a particular brand of commodity and the like.

A **partial correlation coefficient** measures the relationship between any two variables, keeping other variables constant.

The limitations of linear correlations as a technique for the study of economic relations are as follows:

1. The formula for correlation coefficient applies only to linear relationships between variables.
2. That correlation coefficient as a measure of co-variability of variables does not imply any functional relationship between the variables concerned.

## 6.0 TUTOR-MARKED ASSIGNMENT

The following data give the actual sales of a company in each of 8 States of a hypothetical country, together with the forecast of sales by two different methods:



State	Actual Sales (N'million)	Forecast 1	Forecast 2
A	15	13	16
B	19	25	19
C	30	23	26
D	12	26	14
E	58	48	65
F	10	15	19
G	23	28	27
H	17	10	22

- (a) Calculate the rank correlation coefficient between:
- (i) actual sales and forecast 1
  - (ii) actual sales and forecast 2
- (b) Which forecasting method would you recommend next year and why?

## 7.0 REFERENCES/FURTHER READING

- A. Francis (1998). *Business Mathematics and Statistics, 5<sup>th</sup> edition*. Great Britain: Ashford Colour Press.

## MODULE 3

- Unit 1 Forecasting and Time-Series Analysis
- Unit 2 Index Numbers
- Unit 3 Inventory Control
- Unit 4 Decision Analysis
- Unit 5 Network Planning and Analysis

### UNIT 1 FORECASTING AND TIME-SERIES ANALYSIS

#### CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Steps in Forecasting
  - 3.2 Types of Forecasts
  - 3.3 Forecasting Techniques
    - 3.3.1 Moving Averages
    - 3.3.2 Least Squares or Trend Lines
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

#### 1.0 INTRODUCTION

Forecasts are based on past performances. In other words, future values are predicted from past values. This assumes that the future will be basically the same as the past and present, implying that the relationships underlying the phenomenon of interest are stable overtime.

Forecasting can be performed at different levels, depending on the use to which it will be put. Simple guessing, based on previous figures, is occasionally adequate. However, where there is a large investment at stake, structured forecasting is essential.

Any forecasts made, however technical or structured should be treated with caution, since the analysis is based on past data and there could be unknown factors present in the future. However it is often reasonable to assume that patterns that have been identified in the analysis of past data will be broadly continued, at least into the short-term future.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- list the necessary steps in forecasting activities
- enumerate the different types of forecasting
- apply the basic forecasting techniques.

## 3.0 MAIN CONTENT

### 3.1 Steps in Forecasting

We outline the basic steps in forecasting as follows:

- Step 1**      **Gather** past data: daily, weekly, monthly and yearly.
- Step 2**      Adjust or clean up the raw data against inflationary factors. Index numbers can be used in deflating inflationary factors.
- Step 3**      Make forecasts from the “refined” data
- Step 4**      When the future data (which is been forecast) becomes available, compare forecasts with actual values, by so doing, you will be able to establish the error due to forecasting.

### 3.2 Types of Forecasts

The basic types of forecasts are outlined below:

1. Short-term forecasts: these are forecasts concerning the near future. They, are characterised by few uncertainties and therefore more accurate than distant future forecasts
2. Long – term forecasts: these concern the distant future. They are characterised by more uncertainties than short – term forecasts.
3. Extrapolation: these are forecasts based solely on past and present values of the variable to be forecast. In this case, future values are extrapolated from past and present values.
4. Forecasts based on established relationships between the variable to be forecast and other variables.

### 3.3 Forecasting Techniques

The two generally used methods of forecasting include:

1. moving averages
2. trend lines or least squares.

### 3.3.1 Moving Averages

Moving averages can be used to generate the general picture (or trend) behind a set of data or time series. The general pattern generated can be used to forecast future values.

Note that a time series is a name given to numerical data described over a uniform set of time points. Time series occur naturally in all spheres of business activity.

The method of moving averages can be illustrated by the following example.

A monthly sales data is given:

Sales (N)	Jan	Feb	March	April	May	June
Past (Actual)	50	55	70	50	45	90

Using a 3 – period moving averages, the forecast values are:

$$\frac{50 + 55 + 70}{3} = 58 \quad (\text{Feb})$$

$$\frac{55 + 70 + 50}{3} = 58 \quad (\text{March})$$

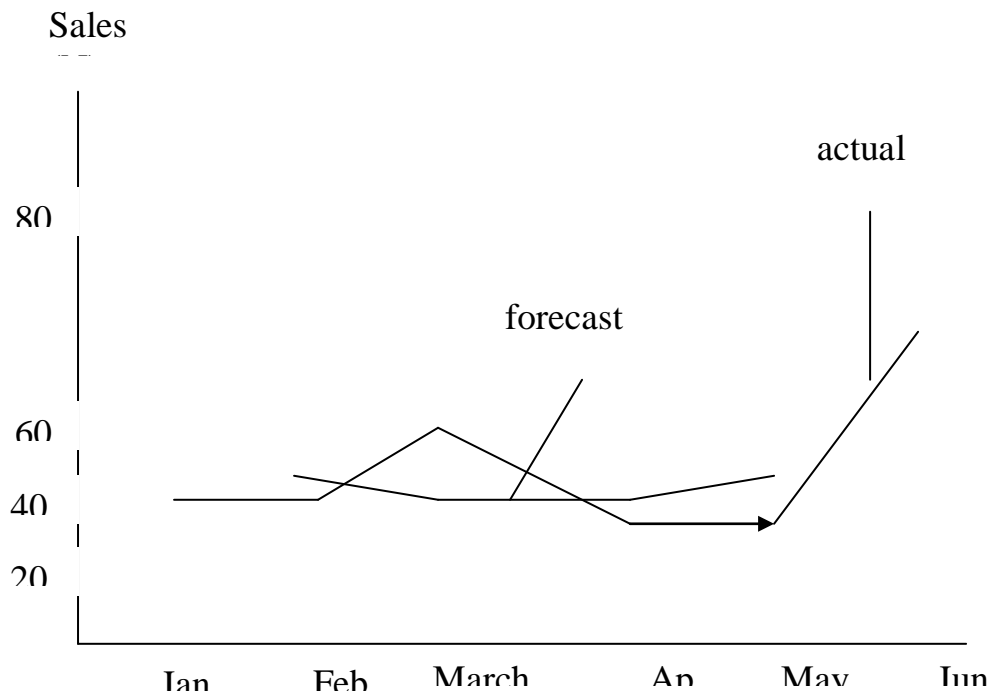
$$\frac{70 + 50 + 45}{3} = 55 \quad (\text{April})$$

$$\frac{50 + 55 + 70}{3} = 58 \quad (\text{May})$$

We can thus summarise the forecast sales as follows:

Forecast sales (N)	Jan	Feb	March	April	May	June
Future (forecast)	-	58	58	55	62	-

These can be presented in a graph as in figure 11.1 below:

**Figure 11.1: Graph of Sales Forecasts.**

### 3.3.2 Least Squares or Trend Lines

The idea behind the use of trend lines in forecasting is based on the assumption that the general picture underlying a given set of data can be reasonably approximated by a straight line. Such a straight line can be extended backwards or forward for predicting past or future values.

#### Example

Suppose the line AB in the following straight line reasonably approximates a set of data for 1995 – 2000

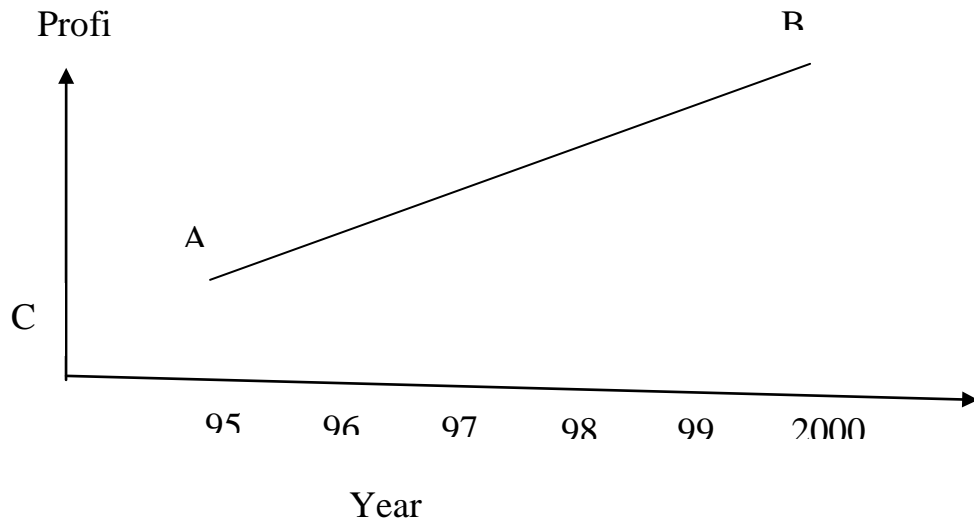
**Figure 11.2: Trend line**

Figure 11.2 indicates that we can forecast profit backwards for years below 1995, using the dotted line AC. Similarly, profits can be forecast for years beyond 2000, using the dotted line BD.

The basic task in using a trend line for forecasting is to determine a line similar to line AB in figure 11.2: then forecasting backwards or forwards is a straight forward activity. The most effective way of determining such a line is the Least-Squares method.

### **The Least – Squares Method**

The least–squares method provides a sound mathematical basis for choosing the best trend line of all possible trend lines for a given set of time series. This method provides an equation (with its numerical coefficients) so that the value corresponding to any given year (or period) can be determined by substituting the given year (or period) into the equation.

For **example**, consider the following periodic data:

**Table 11.1: Time Series Data**

<b>Year (Period)</b> (t)	<b>Output</b> (Y)
1990	50
1991	80
1992	90
1993	49
1994	75
1995	58
1996	82
1997	73
1998	95

With t representing period and Y representing output, the equation showing the relationship between time and output (or the estimated trend line) is given below:

$$\hat{Y} = \hat{a} + b t$$

The Least-Squares method is then used to determine the numerical values of the parameters,  $\hat{a}$  and  $b$

We assume:

- t = 1 in 1990
- t = 2 in 1991
- t = 3 in 1992
- t = 4 in 1993
- t = 5 in 1994
- t = 6 in 1995
- t = 7 in 1996
- t = 8 in 1997
- t = 9 in 1998

Table 11.1 can thus be rewritten as:

<b>t</b>	<b>Y</b>
1	50
2	80
3	90
4	49
5	75
6	58
7	82
8	73
9	95

The formulas for the least – squares estimates are as follows:

$$\hat{a} = \bar{Y} - \hat{b}t$$

Where  $\bar{Y} = \frac{\sum Y}{n}$  ;  $\bar{t} = \frac{\sum t}{n}$  ; n = number of pairs of observations

$$\hat{b} = \frac{n\sum tY - \sum t \sum Y}{n\sum t^2 - (\sum t)^2}$$

Using the given data and second formula, we get:

t	Y	tY	t <sup>2</sup>
1	50	50	1
2	80	160	4
3	90	270	9
4	49	196	16
5	75	375	25
6	58	348	36
7	82	574	49
8	73	584	64
9	95	855	81
45	652	3412	285

Thus,  $n\sum tY = 9(3412) = 30708$

$$\sum t \sum Y = 45(652) = 29340$$

$$n\sum t^2 = 9(285) = 2565$$

$$(n\sum t)^2 = (45)^2 = 2025$$

$$\bar{y} = \frac{\sum Y}{n} = \frac{652}{9} = 72.44$$

$$\bar{t} = \frac{\sum t}{n} = \frac{45}{9} = 5$$

It follows that:

$$\hat{b} = \frac{30708 - 29340}{2562 - 2025} = \frac{1368}{540} = 25.53$$



$$\begin{aligned}
 \hat{a} &= \bar{Y} - b t \\
 &= 72.44 - 2.53 (5) \\
 &= 72.44 - 12.65 \\
 &= 59.79
 \end{aligned}$$

The least – squares line becomes:

$$\hat{Y} = 59.79 + 2.53 t.$$

This equation can be used any time to forecast the value of any given year, provided the numerical value of the year is appropriately identified.

For example, let us forecast the value of output, Y, for year 2003.

Following the systematic process, the year 2003 is associated with the numerical value,  $t = 14$ , so that for  $t = 14$ ,

$$\begin{aligned}
 Y_{2003} &= 59.79 + 2.53 (14) && \text{(by substitution)} \\
 &= 59.79 + 35.4 \\
 &= 95.21
 \end{aligned}$$

Therefore, the forecast value for output in year 2003 is 95.21

### **SELF ASSESSMENT EXERCISE 1**

List the two generally used forecasting techniques and outline the advantages and disadvantages of each.

## **4.0 CONCLUSION**

This unit has exposed you to the basic principles of business forecasting. Forecasts are based on past performances. In other words, future values are predicted from past values. Forecasting can be performed at different levels, depending on the use to which it will be put. Simple guessing, based on previous figures, is occasionally adequate. However, where there is a large investment at stake, structured forecasting is essential.

We were meant to understand that any forecasts made, however technical or structured should be treated with caution, since the analysis is based on past data and there could be unknown factors present in the future. However it is often reasonable to assume that patterns that have

been identified in the analysis of past data will be broadly continued, at least into the short-term future.

## 5.0 SUMMARY

The two generally used methods of forecasting include:

1. moving averages
2. trend lines or least- squares.

Moving averages can be used to generate the general picture (or trend) behind a set of data or time series. The general pattern generated can be used to forecast future values.

The idea behind the use of trend lines in forecasting is based on the assumption that the general picture underlying a given set of data can be reasonably approximated by a straight line. Such a straight line can be extended backwards or forward for predicting past or future values.

The least-squares method provides a sound mathematical basis for choosing the best trend line of all possible trend lines for a given set of time series. This method provides an equation (with its numerical coefficients) so that the value corresponding to any given year (or period) can be determined by substituting the given year (or period) into the equation.

## 6.0 TUTOR-MARKED ASSIGNMENT

Given the following time-series data, with  $t$  representing time period and  $Y$  representing data on a business variable referred to as  $Y$ :

$t$	=	1	2	3	4	5	6	7	8	9	10
$Y$	=	8	11	10	21	4	9	12	10	23	5

- (a) Calculate a set of moving averages of 3 periods
- (b) Using the least-square method, estimate the trend equation for the variable,  $Y$ .

## 7.0 REFERENCES/FURTHER READING

- A. Francis (1998). *Business Mathematics and Statistics, 5<sup>th</sup> edition*. Great Britain: Ashford Colour Press.
- Onwe, O. J. (2007). *Statistical Methods for Business and Economic Decisions: A Practical Approach*. Lagos: Samalice Publishers.

## UNIT 2 INDEX NUMBERS

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- 1.0 Introduction
- 2.0 Objectives
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### 1.0 INTRODUCTION

Index numbers are often used in stabilising the time value of money and in deflating nominal values. An index number measures the percentage change in the value of some economic commodity over a period of time. Index numbers are expressed in terms of a base of 100. Economic commodity is a term used to describe anything measurable and has some economic relevance. Examples include price, wage quantity, productivity, expenditure, and the like.

Two or more time periods are usually involved in the computation of an index number. One of these time periods represents the base time period. The value of the base time period stands as the standard point of comparison, while the values at the other time periods are used to show the percentage in value from the standard value of the base period.

#### Examples

1. Suppose we wish to compare the average monthly salaries of workers in a given company for:

1985, 1990, 1995, 2000, 2005 and 2010, using 1985 as the base year. The salary index for each of the six years can be calculated using the following relationship:

$$I_k = \frac{(\text{Average monthly salary in year K})}{\text{Average monthly salary in 1985}} \times 100$$

The average monthly salaries and their computed index number for each of the six years are presented in table 12.1, referred to as *Index Time Series*.

**Table 12.1 Index Time Series of Monthly Salaries**

Year	Average monthly salaries(N)	Salary Index(1985 = 100)
1985	1,200	100%
1990	2,000	166.7
1995	2,800	233.33
2000	3,600	300
2005	5,000	416.7
2010	6,500	541.7

As can be observed from the above, each wage index is a percentage that indicates the percentage of 1985 salaries that were earned in a given year. For example, the salary index in 1995 indicates that the monthly salary in 1995 was 233.3% of the 1985 monthly salary.

- Suppose the unit price of a standard exercise book was N80 in January and rose to N100 in September, the percentage change in the price of exercise books can be calculated as:

$$\frac{100 - 80}{80} = \frac{20}{80} = 0.25 = 25\%$$

It follows that the price of standard exercise books rose by 25 percent between January and September. Using index number terminology, the 25 percent increase is added to the base price of 100 to get 125. Thus the *price index of standard exercise books in September was 125 (January = 100)*

Note that any increase or decrease must be related to time period, or else it will be meaningless. Index numbers are no exception, hence the (January = 100) as stated above. This:

- (i) gives the starting point (here, January) over which the increase in price is being measured
  - (ii) emphasises the base value (100) of the index number
3. If the productivity of a firm (measured in terms of labour productivity per day) decreased by 10 percent over the periods 2000 to 2006, this 10 percent will be subtracted from 100 to give an index number of 90. This would imply that the productivity index for the company concerned was 90 in the year 2006 (2000 = 100).

At this point, let us look at some index number notations. These notations are useful in the application of the relevant formulas:

$P_o$  = Price at base time point

$P_n$  = Price at some other time point

$Q_o$  = Quantity at base time point

$Q_n$  = Quantity at some other time point

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- describe the term 'indices' and be able to apply them in investment decisions
- state the different classifications of index numbers.

## 3.0 MAIN CONTENT

### 3.1 Index Relatives

An index relative measures the change in a single distinct commodity. A price relative was illustrated in example (2) above, and a productivity relative was computed in example (3) above.

There exists, however more direct ways of calculating index relatives than those demonstrated in examples (2) and (3). In the following formulas we show the method of calculating price and quantity relatives:

Price Relative:  $I_p = P_n/P_o \times 100$

Quantity Relative:  $I_q = Q_n/Q_o \times 100$

### Examples

Table 12.2 below gives details of price and quantities sold of two items (Video and Television) in a particular Cash and Carry Merchandise store over two years (2007 and 2008)

**Table 12.2 Price and Quantity Relatives for 2008**

Item	2007		2008	
	Price $P_o$	Quantity $Q_o$	Price $P_n$	Quantity $Q_n$
Video Recorder	N12,500	40	N12,700	25
Television	N30,000	28	N32,000	40

You are required to calculate the price and quantity relatives for both items for the year 2008 (2007 = 100)

#### Video:

$$\text{Price Relative} = I_{vp} = \frac{P_n}{P_o} \times 100 = \frac{12,700}{12,500} \times 100 = 101.60$$

$$\text{Quantity Relative} = I_{vq} = \frac{Q_n}{Q_o} \times 100 = \frac{25}{40} \times 100 = 62.50$$

#### Television:

$$\text{Price Relative} = I_{tp} = \frac{P_n}{P_o} \times 100 = \frac{32,000}{30,000} \times 100 = 106.67$$

$$\text{Quantity Relative} = I_{tq} = \frac{Q_n}{Q_o} \times 100 = \frac{40}{28} \times 100 = 142.86$$

### 3.3 The Simple Aggregate Price Index

The Simple Price Index can be defined as the ratio of the sum of commodity prices in a given current period to the sum of same commodity prices in the selected base year (or period).

Symbolically,

$$\text{Simple Price Index, PI} = \frac{\sum P_n}{\sum P_o} \times 100,$$

Where  $P_n$  = current year's prices.

$P_o$  = base year's prices.

Note that, given the *Quantity Index (QI)* defined by:

$$QI = \frac{\sum Q_n}{\sum Q_o} \times 100$$

The corresponding *Value Index (VI)* is computed by:

$$VI = \frac{\sum P_n Q_n}{\sum P_o Q_o} \times 100$$

The simple aggregate price index is commonly used in comparing two sets of prices from a wide variety of items.

**Example:**

The average consumer prices, in naira per Kg, for some staple food items in 2000 and 2006 are given in the following table 12.3. We are required to find the Aggregate Price Index for 2006, using 2000 as the base year.

**Table 12.3 Average Consumer Prices for Selected Staple Food Items**

Item	2000	2006
Sugar	10	40
Wheat Flour	11	20
Butter	71	99
Ground beef	91	186
Frying Chicken	39	88

**Solution:**

The required aggregate price index for 2006 is:

$$I_{2006} = \frac{40 + 20 + 99 + 186 + 88}{10 + 11 + 71 + 91 + 39} = \frac{433}{222} \times 100 = 195.06$$

This implies that the prices of the five items are higher by 95.06 percent in 2006 than in 2000.

### 3.4 Weighted Aggregate Price Index

The weighted aggregate price index is the ratio of an aggregate of weighted commodity prices for a given year, K, to an aggregate of the weighted prices of the same commodities in some base year, expressed as a percentage.

Symbolically,

$$I_k = \frac{n \sum_{i=1}^n P_{ki} Q_{ki}}{\sum_{i=1}^n P_{oi} Q_{oi}} \times 100$$

#### Example:

An investor's holding in the shares of four companies in 2004 and 2006 is shown in Table 12.4 below:

**Table 12.4 Investor's Holding of Four Companies**

Company	No of shares		Price per share ( N )	
	2004	2006	2004	2006
A	350	400	0.50	1.25
B	200	180	1.25	3.75
C	140	200	6.25	12.50
D	130	150	12.50	18.75

Using 2004 as base year, calculate:

- a simple aggregate price index and
- a weighted aggregate price index for the investor's holding of shares.

#### Solutions:

- The simple aggregate index is:

$$I_{p2006} = \frac{\sum P_{2006}}{\sum P_{2004}} \times 100 = \frac{400 + 180 + 200 + 150}{350 + 200 + 140 + 130} \times 100 = 1.134 \times 100 = 113.4$$



(b) The weighted aggregate price index is:

$$\begin{aligned}
 I_{wp\ 2006} &= \frac{\sum P_{206} Q_{206}}{\sum P_{204}} \times 100 \\
 &= \frac{1.25(400) + 3.75(180) + 12.50(200) + 18.75(150)}{0.50(350) + 1.25(200) + 6.25(140) + 12.5(130)} \times 100 \\
 &= \frac{500 + 675 + 2500 + 281.5}{175 + 250 + 875 + 1625} \times 100 \\
 &= \frac{6487.5}{2925} = 2.218 \times 100 \\
 &= 221.79
 \end{aligned}$$

### 3.4 The Laspeyres and Paasche Indices

A **Laspeyres index** is a special case of weighted aggregate index. This index uses base time period weights. It is usually associated with price and quantity indices where:

A Laspeyres *Price Index* uses base time period *quantities* as weights

A Laspeyres *Quantity Index* uses base time period *prices* as weights

The formulae for calculating a Laspeyres Index is derived from the general formula for weighted aggregate index. The commonly use formulae are as follows:

$$1. \quad \text{Price Index} = L_p = \frac{\sum P_n Q_o}{\sum P_o Q_o} \times 100$$

$$2. \quad \text{Quantity Index} = L_q = \frac{\sum P_o Q_n}{\sum P_o Q_o} \times 100$$

#### The Paasche Indices

The **Paasche index** is another special case of weighted aggregate indices. It uses current time period weights, and is most commonly associated with price and quantity indices, where:

A Paasche *Price Index* uses current time period *quantities* as weights

A Paasche *Quantity Index* uses current time period *prices* as weights.

As with the Laspeyres index, the formulae most commonly used for calculating a Paasche Index is given as:

$$\text{Price Index} = P_p = \frac{\sum P_n Q_n}{\sum P_o Q_n} \times 100$$

$$\text{Quantity Index } P_q = \frac{\sum P_n Q_n}{\sum P_n Q_o} \times 100$$

### Examples (Paasche and Laspeyres Price Indices):

The data that follow relate to a set of commodities used in a given production process. You are required to compute the Laspeyres and Paasche price indices for the year 2006:

Commodity	Units Purchased	2003 (base Year)		2006	
		Price(P <sub>o</sub> ) (N)	Quantity (Q <sub>o</sub> ) (Units)	Price(P <sub>o</sub> ) (N)	Quantity (Q <sub>o</sub> ) (Units)
A	2 Drum	36	100	40	95
B	1 ton	80	12	90	10
C	10 kg	45	16	41	18
D	100 metres	5	1100	6	1200

### Solutions:

$$\begin{aligned} \text{Laspeyres Price Index} = L_p &= \frac{\sum P_n Q_o}{\sum P_o Q_o} \times 100 \\ &= \frac{40(100) + 90(12) + 41(16) + 6(1100)}{36(100) + 80(12) + 45(16) + 5(1100)} \times 100 \\ &= \frac{12,336}{10,780} \times 100 \\ &= 114.43 \end{aligned}$$

$$\begin{aligned} \text{Paasche Price Index} = P_p &= \frac{\sum P_n Q_n}{\sum P_o Q_n} \times 100 \\ &= \frac{40(95) + 90(10) + 41(18) + 6(1200)}{36(95) + 80(10) + 45(18) + 5(1200)} \times 100 = \frac{12,638}{10,030} \times 100 \\ &= 114.58 \end{aligned}$$

Observe that both of the above indices indicate an average increase in prices of about 14.50 percent. The two indices have slightly different

values because of the differences in the weights used for the commodities.

### 3.5 Comparing Laspeyres with Paasche Price Indices

What the Indices are supposed to calculate.

**Laspeyres:** This compares base period expenditure with a hypothetical current period expenditure at base period quantities.

**Paasche:** compares current period expenditure with a hypothetical base period expenditure at current period quantities when prices are rising.

**Laspeyres Index** tends to over-estimate price increases and the **Paasche index** tends to under-estimate price increases. These are often looked at as disadvantages for these indices.

#### Nature of the Indices

The **Laspeyres Index** is thought of as a 'pure' price index as it compares like with like from period to period. The **Paasche Index** is not considered as 'pure' price index since the weights change from period to period and like is not compared with like. This is the major disadvantage of the Paasche index.

#### Weights used in calculations

Since the *Laspeyres* index uses base period quantities as weights, the index can be outdated. *Paasche index* uses current quantities as weights and hence it is always up-to-date.

#### Ease of calculation

The *Laspeyres index* needs only the base period quantities, no matter the number of periods. The *Paasche index* needs new quantities for each time period.

### 3.6 Uses and Limitations of Index Numbers

#### 3.6.1 Uses of Index Numbers

Index numbers are mostly used to reflect changes in general economic conditions over identified period of time. For instance, the Retail Price Index (RTI) measures changes in the cost of living; the Index of Industrial Production reflects changes in industrial output; the Ordinary Share Index reflects the general state of the Stock Market.

Index numbers are used by policy makers to decide on tax changes, subsidies to parastatals, industries, and so on. Trade Unions use national cost of living and production indices in negotiating for wages or in comparing the cost of living across national boundaries, states, or professions. Insurance companies use various cost indices to index-link house policies.

Other uses of index numbers include:

### 1. Determination of real economic values

In periods of inflation, the actual value of a commodity does not serve as the best guide of its *real value (or worth)*. The worth of any commodity can only be measured relative to the value of associated commodities. Some relevant **indicator** (such as index number) against which to judge value is necessary.

#### Example

Suppose the annual rent of some business shops last year was N250, 000, and this year it has been increased by 10 percent to N275, 000. You can observe that the actual cost is higher. If we obtain some information that the cost of business shops in the state as a whole has risen by 15 percent over the past year, you can rightly argue that the real cost of the given shops has decreased. But if business turnover for the shops (as an alternate indicator) has only increased by 5 percent, you may argue that the real cost of the shops has increased. It follows that, depending on the particular indicator chosen, the real value of a commodity can change.

There are two major types of standard national indicators. These include: *rate of inflation* (often represented by the Retail Price Index), and the *Index of Output of the Production industry*.

### 2. Time-Series Deflation

Time-series deflation is used to obtain a set of index relatives that measure changes in the real value of some commodities with respect to some given indicators. In general, given a time-series with x-values, and some indicator index series with I-values for comparison, the *Real Value Index* (RVI) for period n is determined by:

$$RVI = \frac{\text{Current Value } (x_n)}{\text{Base Value } (x_o)} \times \frac{\text{Base Indicator } (I_o)}{\text{Current Indicator } (I_n)} \times 100$$

$$= \frac{x_n}{x_o} \times \frac{I_o}{I_n} \times 100$$

**Example:**

The following table presents the values of the average daily wages (N) and Relative Price Index for eight months. You are required to compute the monthly real wage indices.

Month	1	2	3	4	5	6	7	8
Daily Wage (N)	17.60	18.10	18.90	19.60	20.25	20.30	20.60	21.40
Retail Price Index	106.1	107.9	112.0	113.1	116.0	117.4	119.5	119.7

Real Wage Index for *Month 1 (Base Period)*:

$$\frac{17.60}{17.60} \times \frac{106.1}{106.1} \times 100 = 100$$

Real Wage Index for *Month 2*:

$$\frac{18.10}{17.60} \times \frac{106.1}{107.9} \times 100 = 100.94$$

Real Wage Index for *Month 3*:

$$\frac{18.90}{17.60} \times \frac{106.1}{112.0} \times 100 = 102.03$$

Real Wage Index for *Month 4*:

$$\frac{19.60}{17.60} \times \frac{106.1}{113.1} \times 100 = 106.6$$

Real Wage Index for *Month 5*:

$$\frac{20.25}{17.60} \times \frac{106.1}{116.0} \times 100 = 105.3$$

Real Wage Index for *Month 6*:

$$\frac{20.30}{17.60} \times \frac{106.1}{117.4} \times 100 = 104.2$$

Real Wage Index for *Month 7*:

$$\frac{20.60}{17.60} \times \frac{106.1}{119.5} \times 100 = 104.2$$

Real Wage Index for *Month 8*:

$$\frac{21.40}{17.60} \times \frac{106.1}{119.7} \times 100 = 108.2$$

The last real wage index indicates that the average daily wage had increased by 8.2 percent over the eight-month period. In real terms, wages increased steadily, except for months 6 and 7.

### 3.6.2 Limitations of Index Numbers

The limitations include, among others:

1. Indices give only general view about economic changes over wide geographical areas. It follows that they will not generally cater for minority groups or professions, or measure regional variations. For instance, an individual's food bill for a particular month may have increased significantly even though the food index or Retail Price Index for that month may have shown a decrease.
2. Weighting factors can become out-dated, thus the index may fail to compare 'like with like'.
3. If samples were used to obtain data for either values or weights, the information obtained might be biased, incomplete or false.
4. Index numbers can be misinterpreted by uninformed laypersons.

### SELF ASSESSMENT EXERCISE 1

Outline the uses of index numbers in business and economic decisions.

## 4.0 CONCLUSION

Index numbers are often used in stabilising the time value of money and in deflating nominal values. An index number measures the percentage change in the value of some economic commodity over a period of time. Index numbers are expressed in terms of a base of 100. Economic commodity is a term used to describe anything measurable and has some economic relevance. Examples include price, wage quantity, productivity, expenditure, and the like.

Two or more time periods are usually involved in the computation of an index number. One of these time periods represents the base time period. The value of the base time period stands as the standard point of comparison, while the values at the other time periods are used to show the percentage in value from the standard value of the base period.

## 5.0 SUMMARY

The Simple Price Index can be defined as the ratio of the sum of commodity prices in a given current period to the sum of same commodity prices in the selected base year (or period).

The weighted aggregate price index is the ratio of an aggregate of weighted commodity prices for a given year, K, to an aggregate of the weighted prices of the same commodities in some base year, expressed as a percentage.

Symbolically,

$$I_k = \frac{\sum_{i=1}^n P_{ki} Q_{ki} \times 100}{\sum_{i=1}^n P_{oi} Q_{oi}}$$

A **Laspeyres index** is a special case of weighted aggregate index. This index uses base time period weights. It is usually associated with price and quantity indices where:

A Laspeyres *Price Index* uses base time period *quantities* as weights

A Laspeyres *Quantity Index* uses base time period *prices* as weights

The **Paasche index** is another special case of weighted aggregate indices. It uses current time period weights, and is most commonly associated with price and quantity indices, where:

A Paasche *Price Index* uses current time period *quantities* as weights

A Paasche *Quantity Index* uses current time period *prices* as weights.



The two basic uses of index numbers are (i) determination of real economic value; and, (ii) time-series deflation.

There are some limitations to the use of index numbers among which are:

- (i) They give only general view about economic changes over wide geographical areas;
- (ii) Weighting factors can become out-dated, thus the index may fail to compare 'like with like';
- (iii) Index numbers can be misinterpreted by uninformed laypersons.

## 6.0 TUTOR-MARKED ASSIGNMENT

The following data present the gross income of a given category of family compared with the Retail Price Index over a seven year period:

Year	1998	1999	2000	2003	2004	2005	2006
Family Income (N'000s)	10,000	12,105	12,809	13,600	13,911	14,001	14,500
Retail Price Index	134.8	157.1	182.0	197.1	223.5	263.7	295.0

- (a) Compute the indices of real gross income (1998 = 100)
- (b) Compute the indices of real gross income using the Retail Price Index as indicator

## 7.0 REFERENCES/FURTHER READING

Onwe, O. J. (2007). *Statistical Methods for Business and Economic Decisions: A Practical Approach* (Lagos: Samalice Publishers)

## UNIT 3 INVENTORY CONTROL

### CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Inventory Costs
  - 3.2 Important Terminologies in the Inventory Control Theories
  - 3.3 Graphical Representation of Inventory Control Problems
  - 3.4 The Inventory Control Systems
    - 3.4.1 Re-order Level System
    - 3.4.2 The Periodic Review System
  - 3.5 Inventory Control Models
    - 3.5.1 The Basic Model
    - 3.5.2 The Adapted Model
- 4.0 Conclusion
- 5.0 Summary
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- 7.0 References/Further Reading

### 1.0 INTRODUCTION

Decisions involving inventories have become an essential part of management decisions. It is important to note that a business firm must know how it can efficiently and effectively manage its inventories if one of its major objectives is business survival.

The aim of an *inventory control system* is to minimise costs and establish:

- (a) the optimum amount of stock to be ordered
- (b) the period between orders

A firm's inventory is defined as the total stocks of various kinds including: basic raw materials; partly finished goods and materials; sub-assemblies; office and workshop supplies and finished goods.

An inventory must be carried by a firm for various reasons:

1. Anticipating normal demand
2. Taking advantage of bulk-purchase discounts
3. Meeting emergency shortages
4. As a natural part of the production process
5. Absorbing wastages and unpredictable fluctuations

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define inventory and explain what inventory control is all about
- analyse the cost associated with inventories
- enumerate the inventory control systems
- apply modern inventory control models.

## 3.0 MAIN CONTENT

### 3.1 Inventory Costs

Like any other business decision variables there are **costs associated with inventories**. These include:

#### 1. Ordering (Replacement) Costs

These are such costs as transportation costs, clerical and administrative costs associated with the physical movement of the purchased external goods. Where the goods are manufactured internally, there are alternative initial costs to be borne with each production run referred to as set-up costs

#### 2. Holding (Carrying) Costs

These are:

- (a) storage costs in terms of staffing, equipment maintenance, and handling;
- (b) storage overheads (heat, light, rent, and the like);
- (c) cost of capital tied up in inventory;
- (d) insurance, security and pilferage;
- (e) deterioration or breakages.

**3. Stock out Costs**

These are costs associated with running out of stock. These include penalty payments, loss of goodwill, idle manpower and machine, and the like.

### 3.2 Important Terminologies in the Inventory Control Theories

The followings are some important terminologies used in inventory control theories. You need to know these to enable you make meanings out of inventory theories.

**Lead Time:** This is the time between ordering of goods and their replenishment. Orders may be internal (requiring a production run) or external.

**Economic Ordering Quantity (EOQ):** This refers to the external order quantity that minimises total inventory costs.

**Economic Batch Quantity (EBQ):** This refers to the size of the internal production run that minimises total inventory costs.

**Safety Stock:** This is a term used to describe the stock held to cover possible deviations in demand or supply during the lead time. It is sometimes referred to as buffer or minimum stock.

**Maximum Stock:** This is a level used as an indicator above which stocks are deemed to be too high.

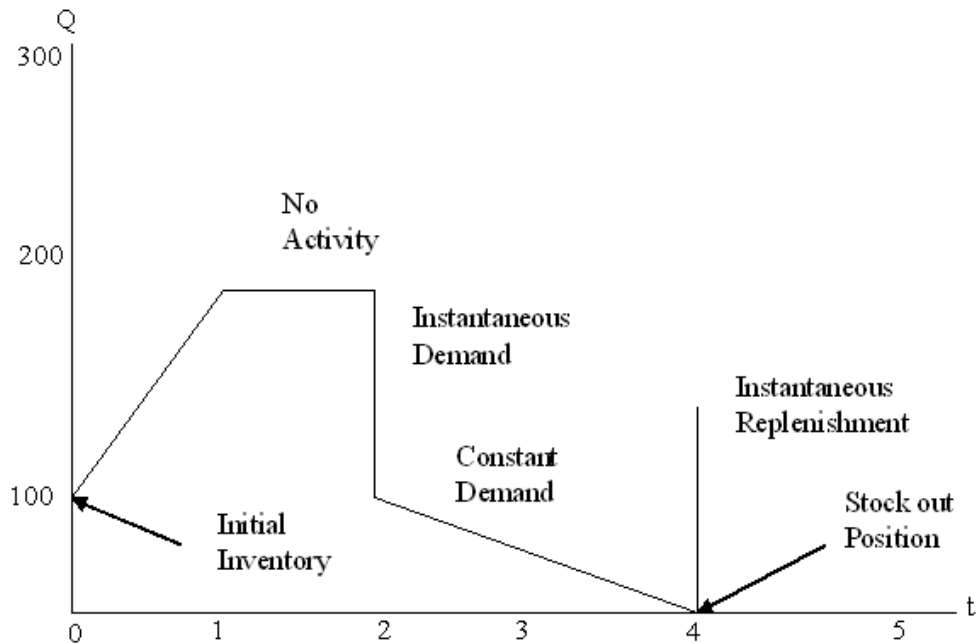
**Reorder Level:** This is the level of stock, which when reached, signals replenishment order.

**Reorder Quantity:** This is the level of replenishment order.

### 3.3 Graphical Representation of Inventory Control Problems

The purpose of inventory graph is to present the inventory control problems in graphical terms. It plots the relationship between quantity of stock held (Q) and time (t).

Figure 13.1 presents a general inventory graph with various features. It shows an initial inventory of 100 items, replenished by a further 100 items continuously over a given time period. Observe as indicated that for the next time period, there was no activity, but at time period 2, 100 items were demanded, followed, over the next two periods, by a continuous demand which used up the last 100 items. This stock out position led to the delivery of additional 150 items.

**Figure 13.1: The General Inventory Graph**

### 3.4 The Inventory Control Systems

The periodic review system sets a review period, at the end of which the stock level of the given item is brought up to a predetermined value. There are two basic inventory control systems.

#### 3.4.1 Re-order Level System

This is the most commonly used control system. It generally results in lower stocks. The system also enables items to be ordered in more economic quantities and is more responsive to fluctuations in demand than the second system discussed below.

The system sets the value of three important levels of stock as warning  $t$  or action triggers for management:

- (i) **Re-order Level:** This is an action level of stock which leads to the replenishment order, normally the Economic Order Quantity (EOQ). For a particular time period, the re-order level is computed as follows:

$$L_{ro} = \text{maximum usage per period} \times \text{maximum lead time (in periods)}$$

- (ii) **Minimum Level:** This is a warning level set such that only in extreme cases (above average demand or late replenishment) should it be breached. It is computed as follows:

$$L_{\min} = \text{Re-Order Level} - (\text{normal Usage} \times \text{Average lead time})$$

- (iii) **Maximum Level:** This is another warning level set such that only in extreme cases (low levels of demand) should it be breached. It is computed by:

$$L_{\max} = \text{Re-order Level} + \text{EOQ} - (\text{Minimum usage} \times \text{Minimum lead time})$$

### Example:

Suppose for a particular inventory, there exists:

- the weekly minimum, normal and maximum usage of 600, 1000, and 1400 respectively;
- the lead time which vary between 4 and 8 weeks (average = 6 weeks); and,
- the normal ordering quantity (EOQ) of 20,000.

It follows that:

$$\text{The Re-order Level } (L_{ro}) = 1400 \times 8 = 11,200 \text{ units}$$

$$\text{Minimum Stock Level } (L_{\min}) = 11,200 - 1000 \times 6 = 5,200 \text{ units}$$

$$\begin{aligned} \text{Maximum Stock Level } (L_{\max}) &= 11,200 + 20,000 - 600 \times 4 \\ &= 28,800 \end{aligned}$$

### 3.4.2 The Periodic Review System

- It enables stock positions to be reviewed periodically so that the chances of obsolete stock items are minimised.
- Economies of scale are possible when many items are ordered at the same time or in the same sequence.

### 3.5 Inventory Control Models

Two basic inventory control models are currently in use. These include:

- The Basic Model
- The Adapted Basic Model (with gradual replenishment)

The models are as discussed below.

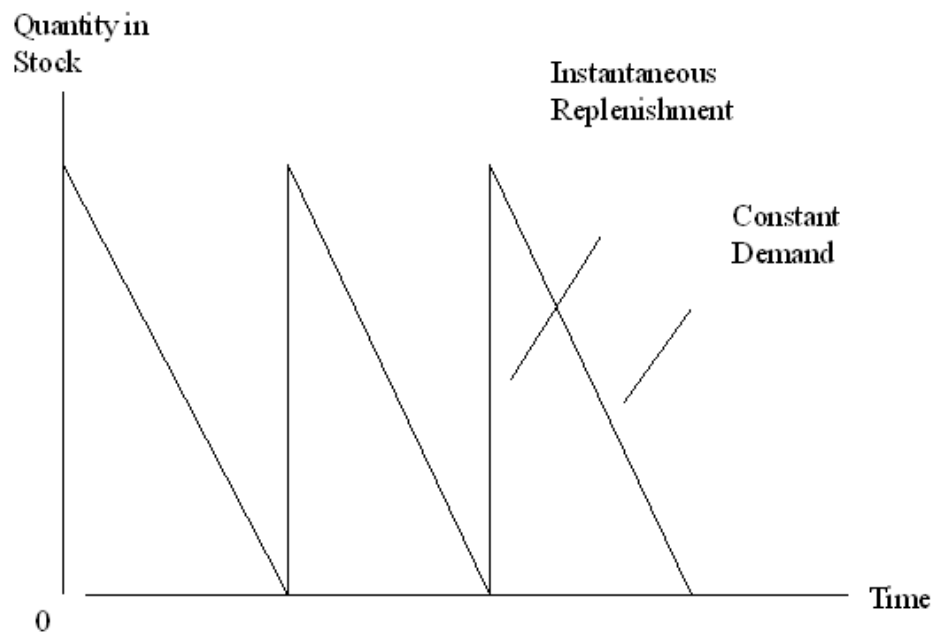
### 3.5.1 The Basic Model

The basic inventory control model is based on the following assumptions:

- (i) The rate of demand (that is, the number of items demanded per year),  $D$ , is constant and continuous over a given period, and no excess demands.
- (ii) The ordering cost ( $C_o = N/\text{circle}$ ) is constant and independent of the quantity ordered.
- (iii) Only one type of stock item is considered and its price ( $P = N/\text{item}$ ) is constant.
- (iv) The holding cost ( $C_h = N/\text{item}$ ) is the cost of carrying one article in stock for one year.
- (v) The quantity ordered per circle ( $q$ ) is supplied to store instantaneously whenever the inventory level becomes zero.

Figure 13.2 illustrates the standard inventory graph for the basic model:

**Figure 13.2: Inventory Graph for the Basic Inventory Control Model**



For the basic model, the total annual inventory cost is minimised when the Economic Ordering Quantity (EOQ) takes the following value:



$$EOQ = \frac{\sqrt{2DC_o}}{C_h}$$

Where D = annual demand

$C_o$  = Order Cost per circle

$C_h$  = Holding Cost per item

The total annual inventory cost, C, is defined by:

C = Total Ordering Cost + Total Holding Cost  
 = number of orders per year x order cost + average inventory level x holding cost per item

$$C = D/q \times C_o + q/2 \times C_h$$

Notice from the above equation that as q gets larger:

Annual ordering cost becomes smaller; and annual holding cost becomes larger.

The basic model also involves the calculation of the following statistics:

(a) Number of orders per year =  $\frac{\text{Yearly Demand}}{\text{EOQ}}$

(b) Length of Cycle (days) =  $\frac{\text{Number of Days per year}}{\text{Number of Orders per year}}$

(c) Average Inventory Level =  $\frac{\text{EOQ}}{2}$

### Examples:

A commodity has a steady rate of demand of 2,000 units per year. Placing an order costs N200 and it costs N50 to hold a unit for a year:

- Estimate the Economic Order Quantity (EOQ)
- Find the number of orders placed per year
- What is the length of the inventory circle?

### Solutions:

Note that from the given information, D = 2,000;  $C_o$  = 200; and,  $C_h$  = 50

- (a) The Economic Order Quantity is determined by:

$$\begin{aligned} \text{EOQ} &= \sqrt{\{2DC_o/C_h\}} = \sqrt{\{2(2000)(200)/50\}} \\ &= \sqrt{16000} = 126.491 \end{aligned}$$

Thus, the economic order quantity is about 127 units.

(b) Number of orders per year =  $\frac{\text{Yearly Demand}}{\text{EOQ}}$

$$= \frac{2000}{126.491} = 15.81$$

Thus, the number of orders per year is approximately 16 orders.

(c) The Cycle length =  $\frac{\text{Number of Days per Year}}{\text{Number of Orders per Year}}$

$$= \frac{365}{15.81} = 23.1$$

The cycle length is about 23 days.

### 3.5.2 The Adapted Model

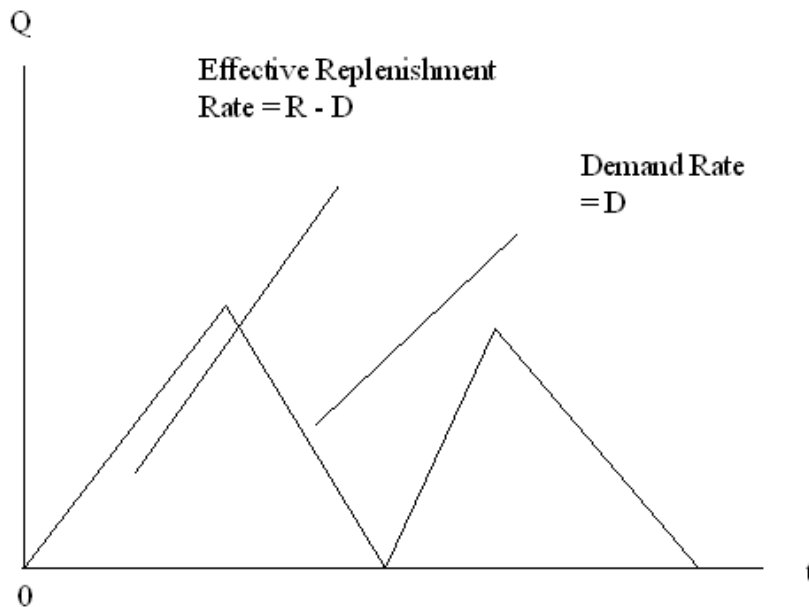
The adapted model assumes gradual replenishment unlike the basic model. This assumption is on the premises that when stocks are received from the production line, it is very likely that finished items are received continuously over a period of time. Stock is therefore, subject to gradual replenishment. Like the basic model, this model also assumes constant demand rate,  $D$ , and the two cost factors: the order cost per cycle,  $C_o$ , and the holding cost per item,  $C_h$ .

The adapted model is often referred to as the *production run* model whereby:

- (i) a production run is started every time the inventory level decreases to zero, and stops when  $q$  items have been produced or supplied. The run lasts for time  $t$  and is known as the *run time*
- (ii) the quantity ordered per cycle is referred to as the *run size*, and items are supplied at the rate of  $R$  per annum
- (iii) the effective replenishment rate is defined as  $R - D$  items.

In figure 13.3, we present the inventory control graph for the adapted model.

**Figure 13.3 The Inventory Control Graph for the Adapted Model**



The value of the run size that minimises inventory cost for the adapted model is defined by:

$$\text{Economic Batch Quantity (EBQ)} = \sqrt{\frac{2DC_o}{C_h \{1 - D/R\}}}$$

Where  $D$  = Annual Demand  
 $R$  = Annual Production rate  
 $C_o$  and  $C_h$  are as defined earlier.

The adapted model uses the following optimal statistics:

- (a) Number of runs per year =  $\frac{\text{Yearly Demand}}{\text{EOQ}}$
- (b) Length of cycle =  $\frac{\text{Number of Days per year}}{\text{Number of Runs per year}}$
- (c) Run time (days) =  $\frac{\text{EBQ} \times \text{Numbers of days per year}}{\text{Annual production rate}}$
- (d) Peak Inventory Level = Effective Replenishment rate  $\times$  run time  
time
- (e) Average inventory level =  $\frac{1}{2} \times$  Peak inventory level

The following **example** illustrates the use of the adapted inventory control model:

### Example

A manufacturing activity requires a continuous supply of 3000 items per year from store, replenished by production runs, each of which operates at the constant rate of 5000 items per year. Each production run has a set-up cost of N300, and the holding cost per item per annum is N25. Compute the Economic Batch Quantity (EBQ), and find the number of runs per year, the length of cycle, the run time, the peak inventory level, and the average inventory level.

### Solutions

From the given information:  $D = 3000$ ;  $R = 5000$ ;  $C_o = 300$ ; and,  $C_h = N25$ .

Thus,

$$\begin{aligned} \text{EBQ} &= \sqrt{\frac{2DC_o}{C_h\{1-D/R\}}} \\ &= \sqrt{\frac{(2 \times 3000 \times 300)}{\{25 \times (1 - 3000/5000)\}}} \\ &= \sqrt{1,200,000} = 1095.445 \end{aligned}$$

It follows that the Economic Batch Quantity (EBQ) is approximately 1096 units

The number of runs per year =  $3000/1096 = 2.74$

The length of cycle =  $365/2.74 = 133.21$  days

The run time =  $\frac{1096 \times 365}{5000} = 80.01$  days.

Peak Inventory Level = Effective Replenishment rate  $(R - D) \times$  Run time  
 $= (5000 - 3000) \times 80.1$   
 $= 160,200$  units

Average inventory level =  $\frac{1}{2} \times 160,200 = 80,100$  units.

## SELF ASSESSMENT EXERCISE 1

1. What is an inventory? Explain some reasons a firm should carry out an inventory
2. What are the differences between EOQ and EBQ?

### 4.0 CONCLUSION

This unit has looked at inventory control as an essential aspect of managerial function. This is the life wire of any firm aspiring for growth, survival, and continuity.

The aim of an *inventory control system* is to minimise costs and establish:

- (a) the optimum amount of stock to be ordered
- (b) the period between orders

In a given business organisation, an inventory is defined as the total stocks of various kinds including: basic raw materials; partly finished goods and materials; sub-assemblies; office and workshop supplies and finished goods.

An inventory must be carried by a firm for various reasons.

1. Anticipating normal demand
2. Taking advantage of bulk-purchase discounts
3. Meeting emergency shortages
4. As a natural part of the production process
5. Absorbing wastages and unpredictable fluctuations

### 5.0 SUMMARY

The discussions in this unit can be summarised as follows:

Just like any other activity in a business setup, the costs associated with inventories include:

- (i) ordering costs;
- (ii) holding costs; and,
- (iii) stock out costs

Two basic inventory systems were identified, including:

- (i) The Re-order Level system; and,
- (ii) The Periodic Review system.

Two basic inventory control models are currently in use. These include:

1. The Basic Model
2. The Adapted Basic Model

## **6.0 TUTOR-MARKED ASSIGNMENT**

An inventory system follows the adapted basic inventory control model. The demand rate is constant at 1,600 items per year, the unit price of the items is N1.80, the holding cost is 10 percent of the unit price per year, and the set-up cost is N12 per run. If the production rate is 5,000 items per year, calculate:

- (a) the optimum run size (EBQ)
- (b) the run time
- (c) the cycle length
- (d) the average inventory level

## **7.0 REFERENCES/FURTHER READING**

Onwe, O. J. (2007) *Statistical Methods for Business and Economic Decisions: A Practical Approach* (Lagos: Samalice Publishers)

## UNIT 4    **DECISION ANALYSIS**

### **CONTENTS**

- 1.0    Introduction
- 2.0    Objectives
- 3.0    Main Content
  - 3.1    Certainty and Uncertainty in Decision Analysis
  - 3.2    Analysis of the Decision Problem
  - 3.3    Expected Monetary Value Decisions
  - 3.4    Decision Making Involving Sample Information
- 4.0    Conclusion
- 5.0    Summary
- 6.0    Tutor-Marked Assignment
- 7.0    References/Further Reading

### **1.0    INTRODUCTION**

Decision analysis is the modern approach to decision making both in economics and in business. It can be defined as the logical and quantitative analysis of all the factors influencing a decision. The analysis forces decision makers to assume some active roles in the decision-making process. By so doing, they rely more on rules that are consistent with their logic and personal behaviour than on the mechanical use of a set of formulas and tabulated probabilities.

The primary aim of decision analysis is to increase the likelihood of good outcomes by making good and effective decisions. A good decision must be consistent with the information and preferences of the decision maker. It follows that decision analysis provides decision-making framework based on available information on the business environment, be it sample information, judgemental information, or a combination of both.

### **2.0    OBJECTIVES**

At the end of this unit, you should be able to:

- identify and analyse decision problems
- analyse decisions that are made under conditions of certainty and uncertainties
- make expected monetary value decisions

### 3.0 MAIN CONTENT

#### 3.1 Certainty and Uncertainty in Decision Analysis

Most decision-making situations involve the choice of one among several alternative actions. The alternative actions and their corresponding payoffs are usually known to the decision-maker in advance. A prospective investor choosing one investment from several alternative investment opportunities, a store owner determining how many of a certain type of commodity to stock, and a company executive making capital-budgeting decisions are some examples of a business decision maker selecting from a multitude of alternatives. The decision maker however, does not know which alternative will be best in each case, unless he/she also knows with certainty the values of the economic variables that affect profit. These economic variables are referred to, in decision analysis, as *states of nature* as they represent different events that may occur, over which the decision maker has no control.

The states of nature in decision problems are generally denoted by  $s_i$  ( $i = 1, 2, 3, \dots, k$ ), where  $k$  is the number of or different states of nature in a given business and economic environment. It is assumed here that the states of nature are mutually exclusive, so that no two states can be in effect at the same time, and collectively exhaustive, so that all possible states are included within the decision analysis.

The alternatives available to the decision maker are denoted by  $a_i$  ( $i = 1, 2, 3, \dots, n$ ), where  $n$  is the number of available alternatives. It is also generally assumed that the alternatives constitute a mutually exclusive, collectively exhaustive set.

When the state of nature,  $s_i$ , whether known or unknown, has no influence on the outcomes of given alternatives, we say that the decision maker is operating under *certainty*. Otherwise, he/she is operating under *uncertainty*.

Decision making *under certainty* appears to be simpler than that under uncertainty. Under certainty, the decision maker simply appraises the outcome of each alternative and selects the one that best meets his/her objective. If the number of alternatives is very high however, even in the absence of uncertainty, the best alternative may be difficult to identify. Consider, for example, the problem of a delivery agent who must make 100 deliveries to different residences scattered over Lagos metropolis. There may literally be thousands of different alternative routes the agent could choose. However, if the agent had only 3 stops to make, he/she could easily find the least-cost route.



Decision making *under uncertainty* is always complicated. It is the probability theory and mathematical expectations that offer tools for establishing logical procedures for selecting the best decision alternatives. Though statistics provides the structure for reaching the decision, the decision maker has to inject his/her intuition and knowledge of the problem into the decision-making framework to arrive at the decision that is both theoretically justifiable and intuitively appealing. A good theoretical framework and commonsense approach are both essential ingredients for decision making under uncertainty.

To understand these concepts, consider an investor wishing to invest N100, 000 in one of three possible investment alternatives, A, B, and C. Investment A is a Savings Plan with returns of 6 percent annual interest. Investment B is a government bond with 4.5 percent annual interest. Investments A and B involve no risks. Investment C consists of shares of mutual fund with a wide diversity of available holdings from the securities market. The annual return from an investment in C depends on the uncertain behaviour of the mutual fund under varying economic conditions.

The investors available actions ( $a_i$ ;  $I = 1, 2, 3, 4$ ) are as follows

- $a_1$ : do not invest
- $a_2$ : select investment A the 6% bank savings plan.
- $a_3$ : select investment B, the 4.5 % government bond.
- $a_4$ : select investment C, the uncertain mutual fund

Observe that actions  $a_1$  to  $a_3$  do not involve uncertainty as the outcomes associated with them do not depend on uncertain market conditions. Observe also that action  $a_2$  dominates actions  $a_1$  and  $a_3$ . In addition, action  $a_1$  is clearly inferior to the risk-free positive growth investment alternatives  $a_2$  and  $a_3$  as it provides for no growth of the principal amount.

Action  $a_4$  is associated with an uncertain outcome that, depending on the state of the economy, may produce either a negative return or a positive return. Thus there exists no apparent dominance relationship between action  $a_4$  and action  $a_2$ , the best among the actions involving no uncertainty.

Suppose the investor believes that if the market is down in the next year, an investment in the mutual fund would lose 10 percent returns; if the market stays the same, the investment would stay the same; and if the market is up, the investment would gain 20 percent returns. The investor has thus defined the states of nature for his/her investment decision-making problem as follows:

- $s_1$ : the market is down.  
 $s_2$ : the market remains unchanged.  
 $s_3$ : the market is up.

A study of the market combined with economic expectations for the coming year may lead the investor to attach subjective probabilities of 0.25, 0.25, and 0.50, respectively, the states of nature,  $s_1$ ,  $s_2$ , and  $s_3$ . The major question is then, how can the investor use the foregoing information regarding investments A, B, and C, and the expected market behaviour serves as an aid in selecting the investment that best satisfies his/her objectives? This question will be considered in the sections that follow.

### 3.2 Analysis of the Decision Problem

In problems involving choices from many alternatives, one must identify all the actions that may be taken and all the states of nature which occurrence may influence decisions. The action to take none of the listed alternatives whose outcome is known with certainty may also be included in the list of actions. Associated with each action is a list of payoffs. If an action does not involve risk, the payoff will be the same no matter which state of nature occurs.

The payoffs associated with each possible outcome in a decision problem should be listed in a *payoff table*, defined as a listing, in tabular form, of the value payoffs associated with all possible actions under every state of nature in a decision problem.

The payoff table is usually displayed in grid form, with the states of nature indicated in the columns and the actions in the rows. If the actions are labeled  $a_1, a_2, \dots, a_n$ , and the states of nature labeled  $s_1, s_2, \dots, s_k$ , a payoff table for a decision problem appears as in table 14.1 below. Note that a payoff is entered in each of the  $nk$  cells of the payoff table, one for the payoff associated with each action under every possible state of nature.

**Table 14.1: The Payoff Table**

ACTION	State of Nature				
	$s_1$	$s_2$	$s_3$	...	$s_k$
$a_1$					
$a_2$					
$a_3$					
.					
.					
.					
$a_n$					

### Example

The managing director of a large manufacturing company is considering three potential locations as sites at which to build a subsidiary plant. To decide which location to select for the subsidiary plant, the managing director will determine the degree to which each location satisfies the company's objectives of minimising transportation costs, minimising the effect of local taxation, and having access to an ample pool of available semi-skilled workers. Construct a payoff table and payoff measures that effectively rank each potential location according to the degree to which each satisfies the company's objectives.

### Solution

Let the three potential locations be sites A, B, and C. To determine a payoff measure to associate with each of the company's objectives under each alternative, the managing director subjectively assigns a rating on a 0 to 10 scale to measure the degree to which each location satisfies the company's objectives. For each objective, a 0 rating indicates complete dissatisfaction, while a 10 rating indicates complete satisfaction. The results are presented in table 14.2 below:

**Table 14.2: Ratings for three alternative plant sites for a Manufacturing Company**

<i>COMPANY OBJECTIVE</i>	<i>ALTERNATIVE</i>		
	Site A	Site B	Site C
Transportation Costs	6	4	10
Taxation Costs	6	9	5
Workforce Pool	7	6	4

To combine the components of payoff, the managing director asks himself, what are the relative measures of importance of the three company objectives I have considered as components of payoff? Suppose the managing director decides that minimising transportation costs is most important and twice as important as either the minimisation of local taxation or the size of workforce available. He/she thus assigns a weight of 2 to the transportation costs and weights of 1 each to taxation costs and workforce. This will give rise to the following payoff measures:

$$\text{Payoff (Site A)} = 6(2) + 6(1) + 7(1) = 25$$

$$\text{Payoff (Site B)} = 4(2) + 9(1) + 6(1) = 23$$

$$\text{Payoff (Site C)} = 10(2) + 5(1) + 4(1) = 29$$

### 3.3 Expected Monetary Value Decisions

A decision-making procedure, which employs both the payoff table and prior probabilities associated with the states of nature to arrive at a decision, is referred to as the *Expected Monetary Value* decision procedure. Note that by *prior probability*, we mean probabilities representing the chances of occurrence of the identifiable states of nature in a decision problem prior to gathering any sample information. The *expected monetary value decision* refers to the selection of available action based on either the expected opportunity loss or the expected profit of the action.

Decision makers are generally interested in the *optimal monetary value decisions*. The optimal expected monetary value decision involves the selection of the action associated with the minimum *expected opportunity loss* or the action associated with the maximum *expected profit*, depending on the objective of the decision maker.

The concept of expected monetary value applies mathematical expectation, where opportunity loss or profit is the random variable and the prior probabilities represent the probability distribution associated with the random variable.

The *expected opportunity loss* is computed by:

$$E(L_i) = \sum_{\text{all } j} L_{ij}P(s_j), \quad (i = 1, 2, \dots, n)$$

Where  $L_{ij}$  is the opportunity loss for selecting action  $a_i$  given that the state of nature,  $s_j$ , occurs and  $P(s_j)$  is the prior probability assigned to the state of nature,  $s_j$ .

The *expected profit for each action is* computed in a similar way:

$$E(\pi_i) = \sum_{\text{all } j} \pi_{ij}P(s_j)$$

Where  $\pi_{ij}$  represents profits for selecting action  $a_i$

#### Example

By recording the daily demand for a perishable commodity over a period of time, a retailer was able to construct the following probability distribution for the daily demand levels:

**Table 14.3: Probability Distribution for the Daily Demand**

$s_i$	$P(s_j)$
1	0.5
2	0.3
3	0.2
4 or more	0.0

The opportunity loss table for this demand-inventory situation is as follows:

**Table 14.4: The Opportunity Loss Table**

Action, Inventory	State of Nature, Demand		
	$s_1(1)$	$s_2(2)$	$s_3(3)$
$a_1(1)$	0	3	6
$a_2(2)$	2	0	3
$a_3(3)$	4	2	0

We are required to find the inventory level that minimises the expected opportunity loss.

**Solution**

Given the prior probabilities in the first table, the expected opportunity loss is computed as follows:

$$E(L_i) = \sum_{j=1}^3 L_{ij}P(s_j), \text{ for each inventory level, } i = 1, 2, 3.$$

The expected opportunity losses at each inventory level become:

$$E(L_1) = 0(0.5) + 3(0.3) + 6(0.2) = \text{N}2.10$$

$$E(L_2) = 2(0.5) + 0(0.3) + 3(0.2) = \text{N}1.60$$

$$E(L_3) = 4(0.5) + 2(0.3) + 0(0.2) = \text{N}2.60$$

It follows that in order to minimise the expected opportunity loss, the retailer should stock 2 units of the perishable commodity. This is the optimal decision.

### 3.4 Decision Making Involving Sample Information

Prior probabilities are acquired either by subjective selection or by computation from historical data. No current information describing the probability of occurrence of the states of nature was assumed to be available.

In many cases, observational information or other evidences are available to the decision maker either for purchase or at the cost of experimentation. For example, a retailer whose business depends on the weather may consult a meteorologist before making decisions, or an investor may hire a market consultant before investing. Market surveys carried out before the release of a new product represent another area in which the decision maker may seek additional information. In each of these examples, the decision maker attempts to acquire information relative to the occurrence of the states of nature from a source other than that from which the prior probabilities were computed.

When such information is available, *Baye's Law* can be employed to revise the prior probabilities to reflect the new information. These revised probabilities are referred to as *posterior probabilities*.

By definition, the *posterior probability* represented symbolically by  $P(s_k/x)$  is the probability of occurrence of the state of nature  $s_k$ , given the sample information,  $x$ . This probability is computed by:

$$P(s_k/x) = \frac{P(x/s_k)P(s_k)}{\sum_{\text{all } i} P(x/s_i)P(s_i)}$$

The probabilities,  $P(x/s_i)$  are the conditional probabilities of observing the observational information,  $x$ , under the states of nature,  $s_i$ , and the probabilities  $P(s_i)$  are the prior probabilities.

The expected monetary value decisions are formulated in the same way as before, except that the posterior probabilities are used instead of prior probabilities. If the objective is to minimise the expected opportunity loss, the quantity is computed for each action . The expected opportunity loss in this case is computed by:

$$E(L_i) = \sum_{\text{all } j} L_{ij}P(s_j/x) \quad I = 1, 2, 3, \dots, n$$

#### Example

It is known that an assembly machine operates at a 5 percent or 10 percent defective rate. When running at a 10 percent defective rate, the machine is said to be out of control. It is then shut down and readjusted.

From past experience, the machine is known to run at 5 percent defective rate 90 percent of the time. A sample of size  $n = 20$  has been selected from the output of the machine, and  $y = 2$  defectives have been observed. Based on both the prior and sample information, what is the probability that the assembly machine is in control (running at 5 percent defective rate)?

### Solution

The states of nature in this example relates to the assembly machine defective rates. Thus the states of nature include:

$s_1 = 0.05$ , and  $s_2 = 0.10$  with the assumed prior probabilities of occurrence of 0.90 and 0.10. We are required to use these prior probabilities, in line with the observed sample information, to find the posterior probability associated with the state of nature,  $s_1$ .

In this problem, the “experimental information,  $x$ ” is the observation of  $y = 2$  defectives from a sample of  $n = 20$  items selected from the output of the assembly machine. We need to find the probability that the experimental information,  $x$ , could arise under each state of nature,  $s_i$ . This can be done by referring to the binomial probability distribution table found in the appendix.

Under the state of nature  $s_1 = 0.05$ , we obtain:

$P(x/0.05) = P(n = 20, y = 2/0.05) = 0.925 - 0.736 = 0.189$  (from the binomial distribution table)

Under the state of nature,  $s_2 = 0.10$ , we obtain:

$P(x/0.10) = P(n = 20, y = 2/0.10) = 0.677 - 0.392 = 0.285$  (from the binomial distribution table).

We now employ the Baye’s Law to find the posterior probability that the machine is in control ( $s_1$ ) based on both the prior and experimental information. To make the work easy, we use the Columnar’s approach to the use of Baye’s Law as illustrated below:

**Table 14.5: Columnar Approach to Use of Baye's Law**

	(1) State of Nature, $s_i$	(2) Prior, $P(s_i)$	(3) Experimental Information, $P(x/s_i)$	(4) Product, $P(s_i)P(x/s_i)$	(5) Posterior, $P(s_i/x)$
$s_1$	0.05	0.90	0.189	0.1701	0.86
$s_2$	0.10	<u>0.10</u>	0.285	<u>0.0285</u>	<u>0.14</u>
		1.00		0.1986	1.00

Looking at column (4), we observe the product of the entries in columns (2) and (3). These values measure the *joint probabilities*. The sum of the entries in column (4) is the term in the denominator of the formula for Baye's Law and measures the *marginal probability* of observing the experimental information,  $x$ . The posterior probabilities, column (5), are obtained by taking each entry in column (4) and dividing by the sum of the entries in column (4).

Even though we found that 10 percent of the items in the sample is defective (that is, 2 out of the 20 items is defective), the posterior probability that the machine is running at the 10 percent defective rate (running out of control) is only 0.14, which is a little greater than the prior probability that the machine is out of control (0.10). It follows that the probability that the machine is not running out of control is 0.86.

### SELF ASSESSMENT EXERCISE 1

Give the justification for using an expected monetary value objective in decision problems.

## 4.0 CONCLUSION

This unit has uncovered the fact that decision analysis is the modern approach to decision making both in economics and in business. It can be defined as the logical and quantitative analysis of all the factors influencing a decision. Decision analysis forces decision makers to assume some active roles in the decision-making process. By so doing, they rely more on rules that are consistent with their logic and personal behaviour than on the mechanical use of a set of formulas and tabulated probabilities.

The primary aim of decision analysis is to increase the likelihood of good outcomes by making good and effective decisions. A good decision must be consistent with the information and preferences of the decision maker. It follows that decision analysis provides decision-making framework based on available information on the business



environment, be it sample information, judgemental information, or a combination of both.

## 5.0 SUMMARY

The unit considered the following important elements of decision analysis:

### 1. Decision Problems and Analysis

In problems involving choices from many alternatives, one must identify all the actions that may be taken and all the states of nature whose occurrence may influence decisions. The action to take none of the listed alternatives whose outcome is known with certainty may also be included in the list of actions. Associated with each action is a list of payoffs. If an action does not involve risk, the payoff will be the same no matter which state of nature occurs.

### 2. Expected Monetary Value Decisions

A decision-making procedure, which employs both the payoff table and prior probabilities associated with the states of nature to arrive at a decision, is referred to as the *Expected Monetary Value* decision procedure. Note that by *prior probability* we mean probabilities representing the chances of occurrence of the identifiable states of nature in a decision problem prior to gathering any sample information. The *expected monetary value decision* refers to the selection of available action based on either the expected opportunity loss or the expected profit of the action.

### 3. Decision Making Involving Sample Information

In many decisions, observational information or other evidence are available to the decision maker either for purchase or at the cost of experimentation. For example, a retailer whose business depends on the weather may consult a meteorologist before making decisions, or an investor may hire a market consultant before investing. Market surveys carried out before the release of a new product represent another area in which the decision maker may seek additional information. In each of these examples, the decision maker attempts to acquire information relative to the occurrence of the states of nature from a source other than that from which the prior probabilities were computed.

When such information is available, *Baye's Law* can be employed to revise the prior probabilities to reflect the new information. We refer to such probabilities as *posterior probabilities*.

## 6.0 TUTOR-MARKED ASSIGNMENT

A special commission appointed by the governor of Enugu state is currently studying the issue of site location for a nuclear power-generating plant. The alternatives being currently considered are ( $a_1$ ) to locate the plant at the mouth of Ever-Valley River, ( $a_2$ ) to locate the plant inland, and ( $a_3$ ) not to build the plant at all. Because of the heat generated in cooling operations,  $a_1$  may prove damaging to a nearby fishing industry vital to the economy of Enugu state. However, there may be insufficient water flow available at site  $a_2$  for adequate cooling. If alternative  $a_3$  is chosen, sufficient energy resources may not be available to meet future needs.

- (a) If you were the economic adviser to the commission, what would you recommend to be used as a measure of payoff for evaluating the three alternatives?
- (b) Suppose the commission has decided that the important attributes of value in the decision model are
  - (i) the ability of each alternative to meet future commercial power needs,
  - (ii) the ability of each alternative to meet future residential power needs,
  - (iii) the effects on the state's fishing industry, and
  - (iv) other environmental effects. What recommendation would you give to the commission regarding the incorporation of these four attributes into a model to derive some reasonable measure of payoff associated with each alternative?

## 7.0 REFERENCES/FURTHER READING

- A. Francis (1998). *Business Mathematics and Statistics, 5<sup>th</sup> edition* (Great Britain: Ashford Colour Press).

## UNIT 5 NETWORK PLANNING AND ANALYSIS

### CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Activities and Events
  - 3.2 Summary of Network Planning Processes
  - 3.3 Critical Paths
    - 3.3.1 Earliest and Latest Event Times Approach to Critical Path
    - 3.3.2 Tabular Presentation of a Network
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

### 1.0 INTRODUCTION

This unit recognises the managerial importance of networking in production, distribution, and related business activities. *Network planning* is basically the task of breaking down a given business project into its constituent activities and deciding on their time relationships. It involves listing the activities needed to complete a project, as well as defining the time for the completion, organising the order of completion of the activities, and drawing the associated network. This task is very valuable as it can help management in the establishment of objectives and in pinpointing the courses of action to follow. The resulting network chart forms a policy statement.

*Network analysis* covers the way large projects are planned. Such projects include a major construction work, research and development projects, the computerisation of business activities, and the like.

### 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define activities and events
- construct activity networks
- construct precedence tables
- state the network planning processes
- explain the critical path analysis.

### 3.0 MAIN CONTENT

#### 3.1 Activities and Events

An *activity* in this context is either any task requiring effort (such as drawing a plan for weekly production) or simply waiting for something to happen (such as waiting for a delivery of an order).

The choice of a set of activities for a given project depends on the level of analysis required.

An *event* on the other hand, signifies a point in time in the progress of the given project. All activities must start and end with an event.

The following diagram illustrates the concepts of an activity and an event as presented here.

**Figure 15.1: Activities and Events**



In constructing a network of activities and events, you should label *activities* with capital letters and *events* with numbers, as illustrated by figure 15.2 below:

**Figure 15.2: Construction of Network of Activities and Events**

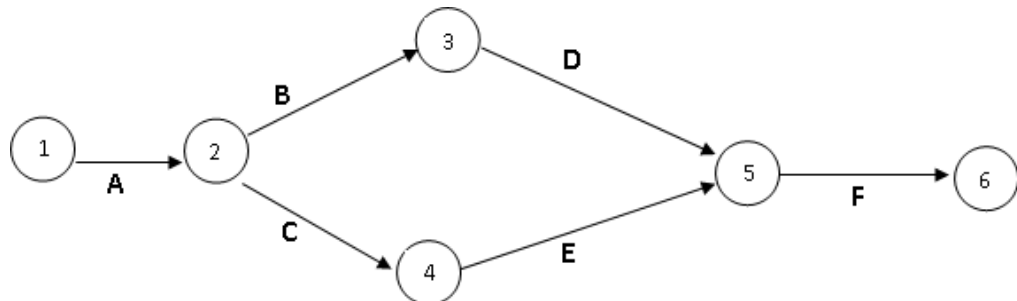


Figure 15.2 illustrates as follows:

- (a) Activities B and C cannot begin until activity A is completed;

- (b) Activity F cannot start until both activities D and E are completed; and,
- (c) It is always the case that there is just one start and end event.

You have to note that before constructing a network, you should construct an activity list showing, for each activity, the preceding or previous activity. This list should be presented in a table referred to as **Precedence Table** as illustrated below (describing the simple network shown in figure 15.2 above):

**Table 15.1: Precedence Table of Network Activities**

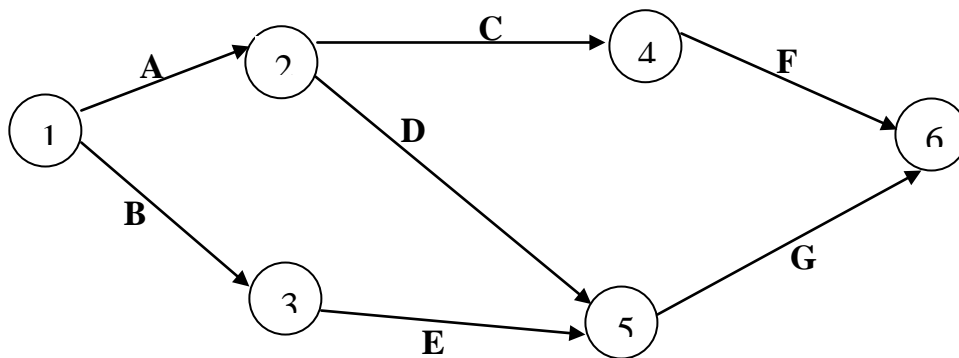
<i>Activity</i>	<i>Precedence Activity</i>
A	-
B, C	A
D	B
E	C
F	D, E

**Other Example**

Construct the network for the following activity list:

<i>Activity</i>	<i>Precedence Activity</i>
A, B	-
C, D	A
E	B
F	C
G	D, E

**Solution:**



In order to ensure that no two activities have the same start and end events you are advised to use what is called **Dummy Activities**.

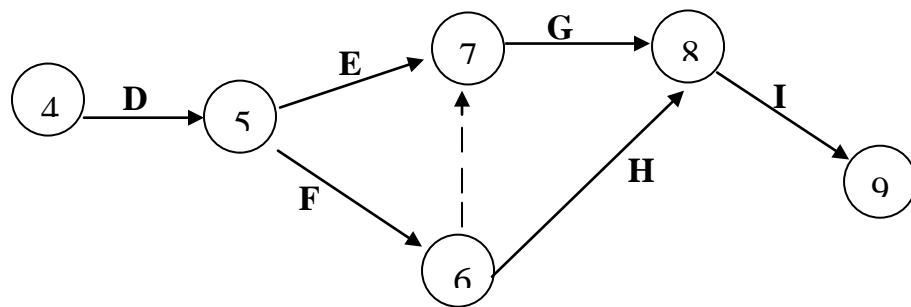
Dummy activities are shown in the network as *dashed* lines, as illustrated by the following example.

**Example**

Construct a network for the following activity list:

Activity	Preceding Activity
---	---
E, F	D
H	F
G	E, F
I	G, H
---	---

**Solution:**



**3.2 Summary of Network Planning Processes**

The network planning process can be summarised by the following steps.

**Step 1:** List the relevant activities and times needed to complete them.

**Step 2:** Decide on the activity order and precedents.

**Step 3:** Sketch the network until it is correct.

**Step 4:** Perform checks in the following way.

Note that there cannot be two activities with the same start and end events. To avoid this, insert *dummy* activities.

Ensure that there is just **one start event** and just **one end event**.

As an example, consider the following description of the activities and the times needed to complete each:

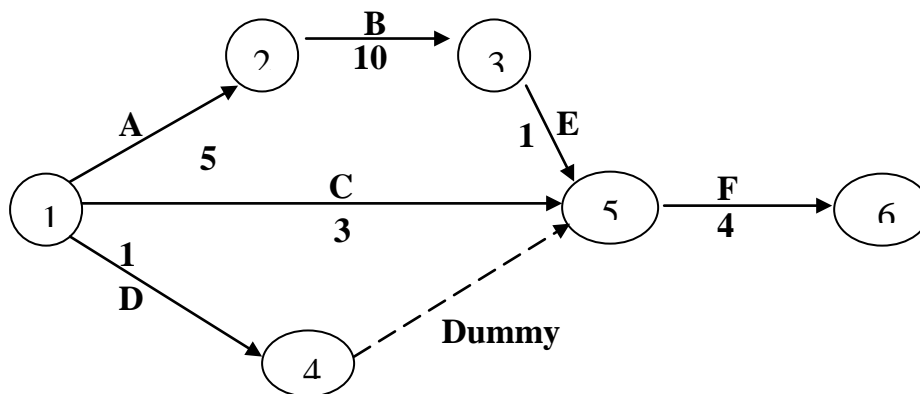
- A: letter dictated to secretary (5 minutes)
- B: letter typed (10 minutes)
- C: envelope addressed (3 minutes)
- D: envelope stamped (1 minute)
- E: letter is put in envelope and sealed (1 minute)

The network plan, including the timings can be illustrated as follows:

### Steps 1 and 2: Activity List, Order and Precedents:

Activity	Preceding Activity
A, C, D	-
B	A
E	B
F	C, D, E

### Steps 3 and 4: Sketch and Checks:



**Note:** The associated times are the highlighted figures.

### 3.3 Critical Paths

If you know the times for all activities for your chosen project, you can evaluate the time taken for each of the possible paths through the network. The longest of these paths is referred to as the **critical path**. The *shortest time* within which the project can be completed is the *total time on this path*.

Consider again the simple network as illustrated above. You will notice there are three possible paths, and for each of these paths, the length of time is as shown below:

<i>Path</i>	<i>Length of Time</i>	<i>Total</i>
ABEF	$5 + 10 + 1 + 4$	<b>20</b>
CF	$3 + 4$	<b>7</b>
D()F	$1 + 4$	<b>5</b>

Note that () represents the dummy activity.

You can observe for the above illustration that the **largest value is 20**. This implies that the **critical path is ABEF**.

It follows that in all critical path problems, if you can identify all the paths through a network, with the times associated with them, then the *path with the longest time* is referred to as *the critical path*.

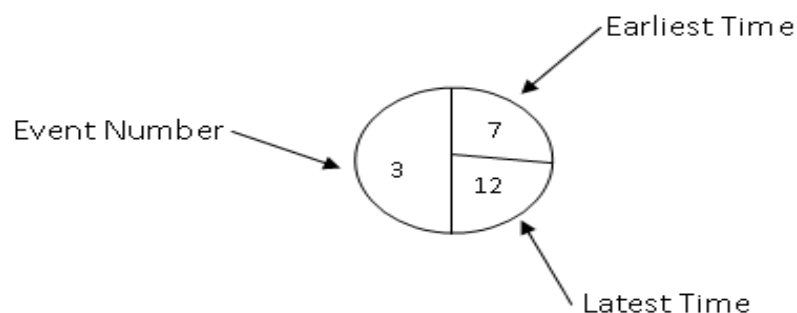
The critical path does not have to be unique and in order to shorten the length of your project, you must reduce the duration time of the activity on the critical path. Reduction of the duration time of an activity outside the critical path will definitely not affect the total time allocated to the project.

### 3.3.1 Earliest and Latest Event Times Approach to Critical Path

A better systematic method of identifying the critical path, suitable for computerisation, involves the calculation of earliest and latest times for events. An example will demonstrate the method employed for this calculation.

Using the notation for marking events presented in figure 15.3, consider the network in figure 15.4 below:

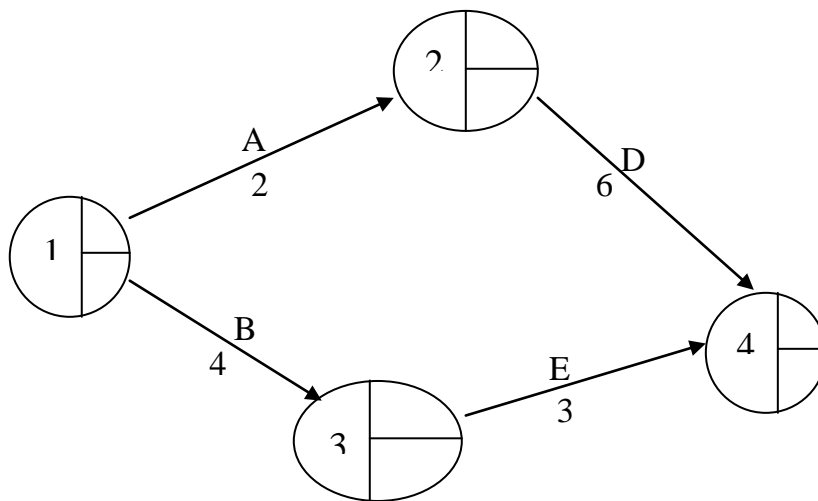
**Figure 15.3: Notation for Marking Events**





**Figure 15.4:** Calculation of earliest and latest times for events

**Figure 15.4.1: Original Network**



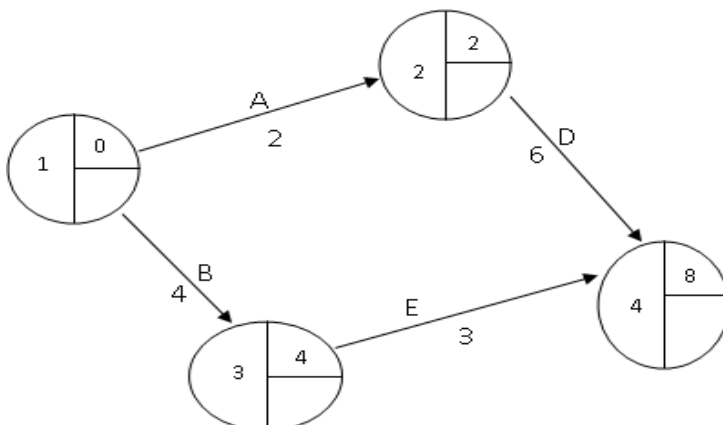
To enter the earliest time, you work from *left to right* in the original network and enter '0' in event 1 early.

The basic technique is to simply go to each event in turn, adding the activity time to the previous event early time. For example, event 2 early =  $E(2) = E(1) + A = 0 + 2 = 2$ . You need to note that in this process, you should follow the *arrow head*, and if there is more than one arrow head leading to an event, you should *take the largest of the possible early times*.

Completing the process,  $E(3) = E(1) + B = 0 + 4 = 4$ ;  
 $E(4) = E(2) + D = 2 + 6 = 8$

The earliest time entered is presented below:

**Figure 15.4.2: The earliest times entered**



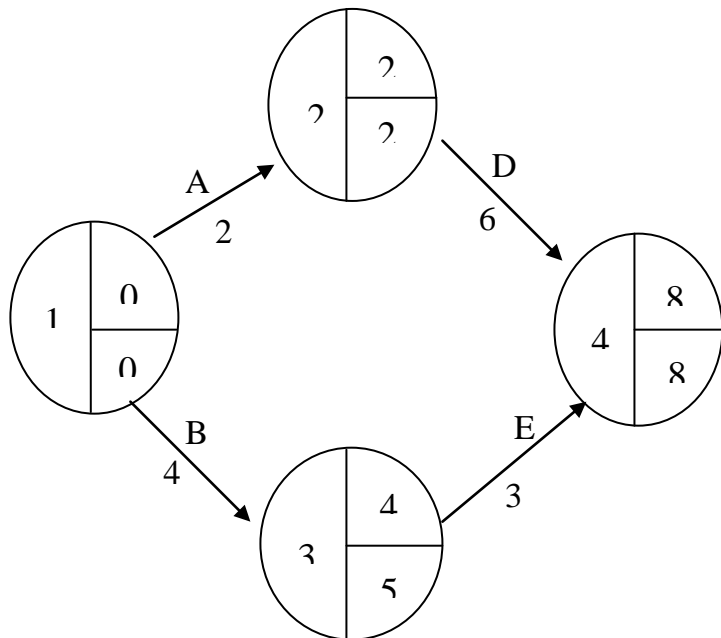
You can now work backwards starting from the final event, event E (4), and fill in all the latest event times.

Start with the *event time* for the last event,  $E(4) = 8$ .

You can now calculate the event late time =  $L(\text{event})$  for each event by subtracting the activity time from  $L(\text{previous})$ . Thus, for the event before event 4,  $E(4)$  which is event 3 or  $E(3)$ , the event late time will be:  $L(3) = L(4) - E = 8 - 3 = 5$ . For events with two possibilities (or two arrow heads, the rule will be the reverse of the earlier discussed event choice circle; if there are more than one tail or arrow head leading from the event, you would have to take the smallest difference. This happens to be the case for event 2.

For the event 2, you can observe the choice of  $8 - 6 = 2$  or  $5 - 1 = 4$ . It follows that by this rule,  $L(2) = 2$ . These calculations are presented in figure 13.4.3 below:

**Figure 15.4.3: Earliest and Latest Times Entered**



To summarise these discussions, note the followings:

**Earliest (E) Calculation:**  $E(\text{event}) = E(\text{previous event}) + \text{Activity Time}$

If there are more than one arrow heads leading to an event, choose the largest event time.

**Latest (L) Calculation:**  $L(\text{event}) = L(\text{next event}) - \text{Activity Time}$   
 If there are 'tail' alternatives, choose the smallest event time.

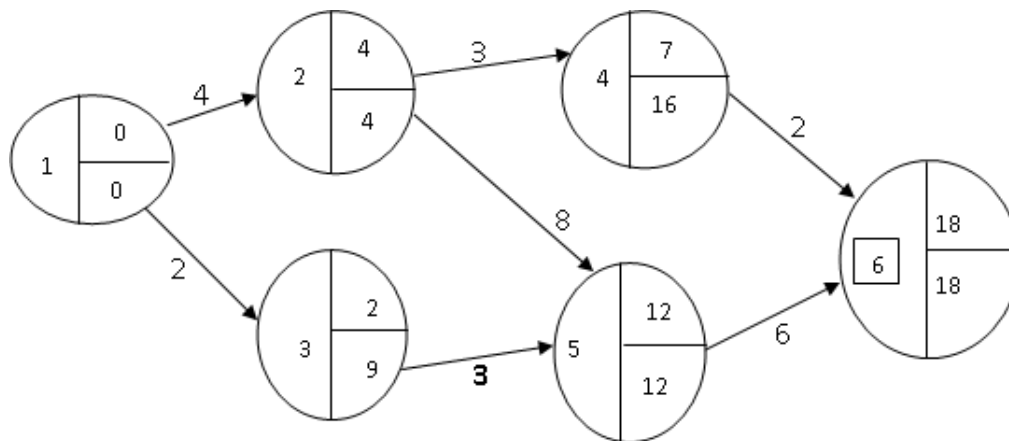
### 3.4.2 Tabular Presentation of a Network

Activity networks can be presented in a tabular form, showing activity name, duration, and earliest and latest start and finish times. In the following example, you will learn how the tabular presentation of networks can be done. As a guide, note first the following relationships:

1. Earliest activity start time = Earliest start of beginning event
2. Earliest activity finish time = Earliest start of beginning event + activity duration
3. Latest activity start time = Latest start of ending event - activity duration
4. Latest activity finish time = Latest start of ending event

#### The Example

You are required to put in a tabular form the following network



#### Solution

Activity	Duration	Start		Finish	
		E	L	E	L
1 - 2	4	0	0	4	4
1 - 3	2	0	7	2	9
2 - 4	3	4	13	7	16
2 - 5	8	4	4	12	12
3 - 5	3	2	9	5	12
4 - 6	2	7	16	9	18
5 - 6	6	12	12	18	18

## SELF ASSESSMENT EXERCISE 1

1. Define activities and event as used in network analysis
2. What is a precedence table?
2. Why are dummy activities necessary?
3. What do we mean by a critical path through a network?

## 4.0 CONCLUSION

This unit has discussed at length the basic principles of activity networking. You have been exposed to the following very important issues.

- 1) Activity and events;
- 2) Construction of network of activities and events;
- 3) Meaning and construction of precedence table;
- 4) Network planning processes; and,
- 5) The critical path analysis

## 5.0 SUMMARY

The importance of networking in production cannot be over emphasised. As pointed out in this unit, network planning appears to be basically the task of breaking down a given business project into its constituent activities and deciding on their time relationships. It involves listing the activities needed to complete a project, as well as defining the time for the completion, organising the order of completion of the activities, and drawing the associated network. This task is very valuable as it can help management in the establishment of objectives and in pinpointing the courses of action to follow. The resulting network chart forms a policy statement.

*Network analysis* in itself covers the way large projects are planned. Such projects include a major construction work, research and development projects, the computerisation of business activities, and the like.

Other important management issues discussed have to do with critical path analysis. We noted that, if you know the times for all activities for your chosen project, you can evaluate the time taken for each of the possible paths through the network. The longest of these paths is referred to as the *critical path*. The *shortest time* within which the project can be completed is the *total time on this path*.

## 6.0 TUTOR-MARKED ASSIGNMENT

The following activities describe the project of 'changing a flat tyre'. The bracketed figures are timing in minutes.

- A = take jack and tools from the boot (1)
- B = remove wheel cover (0.5)
- C = loosen wheel nuts (1.5)
- D = place jack under car ((0.5)
- E = jack up the car (0.5)
- F = get spare tyre from the boot (0.40)
- G = remove wheel nuts and the wheel (0.5)
- H = place spare tyre onto the studs (0.25)
- I = guide nuts onto studs (0.15)
- J = lower car and remove jack (0.25)
- K = replace jack in boot (0.15)
- L = tighten nuts on studs (0.25)
- M = put old wheel and tools back to the boot (1)
- N = replace wheel cover (0.5)

- (a) Draw up the relevant precedence table
- (b) Draw up the network
- (c) Draw up a table showing duration, earliest and latest activity times.

## 7.0 REFERENCES/FURTHER READING

- A. Francis (1998). *Business Mathematics and Statistics, 5<sup>th</sup> edition* (Great Britain: Ashford Colour Press)

**MODULE 4**

- Unit 1 Arithmetic and Geometric Progression
- Unit 2 Interest Rate and Depreciation
- Unit 3 Present Values and Investment Analysis

**UNIT 1 ARITHMETIC AND GEOMETRIC PROGRESSION**

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Progressions
  - 3.2 Arithmetic Progressions (AP)
  - 3.3 Geometric Progression (GP)
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

**1.0 INTRODUCTION**

As a preliminary, this unit begins with discussions on sequences of numbers referred to as arithmetic and geometric progressions. The form of each of these, including the formulae for calculating the totals are also discussed. The important practical business applications of progression will be presented for your future adaptation to investment decisions.

**2.0 OBJECTIVES**

At the end of this unit, you should be able to:

- interpret the concepts of arithmetic and geometric progressions
- explain the computational formulae for arithmetic and geometric progression
- apply the concepts of progressions in daily business and investment decisions.

### 3.0 MAIN CONTENT

#### 3.1 Progressions

Mathematical progressions that are often used in business and economics are those of Arithmetic and Geometric progressions. These feature mostly in mathematics of finance, where they are used in calculating interest earnings on investments. Taking these progressions one at a time, discuss as follows.

#### 3.2 Arithmetic Progressions (AP)

An arithmetic progression refers to a sequence of numbers, called *terms*, in which any number after the first can be obtained from the immediate past number by adding a constant number, called the **common difference, d**.

The sequence of numbers: 1, 5, 9, 13, 17, 21, for example is an arithmetic progression, since the common difference between one term and the next is 4 ( $5 - 1 = 4$ ;  $9 - 5 = 4$ ;  $13 - 9 = 4$ , etc.). The constant difference, 4, is the common difference,  $d$ .

The first term of an arithmetic progression can be denoted by **a**. This implies that an arithmetic progression takes the general form:

$$a, a + d, a + 2d, a + 3d, a + 4d \dots$$

The value of a particular term, the  $n$ -th term, and sum of the terms in an arithmetic progression can be ascertained using the following formulae:

1.  $n$ -th term =  $a_n = a + (n - 1) d$ ,

Where **a** refers to the first-term; **n** refers to the number of terms in the sequence; and **d** refers to the common difference.

2. Sum of the **n** terms in the sequence =  $S_n = n/2 [2a + (n - 1) d]$

#### Examples

Consider the progression: 10, 15, 20, 25 ... This is an arithmetic progression (AP) as each term is obtained by adding 5 to the previous term. The first term in this progression is 10 and the common difference,  $d = 5$ .

To find the value of the 6<sup>th</sup> term,  $a_6$ , and the sum of the first 8 terms of this progression, you will do the following:

First, identify the progression (AP), with first term,  $a = 10$  and the common difference,  $d = 5$ . From the formula,

$a_n = a + (n - 1)d$ , you get:

6<sup>th</sup>- term =  $a_6 = 10 + (6 - 1) (5)$ , since  $n = 6$ .

$$a_6 = 10 + 25 = 35.$$

It follows that the 6<sup>th</sup> term of this progression has the value of 35.

You will obtain the sum of the first 8 terms of this progression using the stated formula for the sum of  $n$  terms (that is, 8 terms in this case):

$$\begin{aligned} S_n = S_8 &= 8/2 [2a + (n - 1) d] \\ &= 4[2(10) + (8 - 1) (5)] \\ &= 4[20 + 35] \\ &= 4(55) \\ &= 220. \end{aligned}$$

This means that the sum of the first 8 terms of the progression in question is 220.

### 3.3 Geometric Progression (GP)

A Geometric Progression (GP) defines a sequence of numbers, referred to as **terms**, in which any term after the first can be obtained from its immediate past term or predecessor by multiplying this past term by a constant term, called the **common ratio** ( $r$ ). Like the arithmetic progression, the first term of the geometric progression is denoted by the letter  $a$ .

For instance, the sequence of the numbers: 5, 15, 45, 135 ... is a geometric progression since each term is being multiplied by the same constant term (3 in this case) to obtain the next term in the sequence. Notice that you can obtain this constant term (that is 3) by dividing the new term by its immediate past term or predecessor:  $15/5 = 3$ ;  $45/15 = 3$ ; and  $135/45 = 3$ . This constant term is the common ratio denoted by  $r$ .

The general form of Geometric Progression can be presented as follows:

$$GP = a, ar, ar^2, ar^3, ar^4 \dots$$



In our example above,  $a = 5$  and  $r = 3$ , given the geometric progression as:

$$5, 5(3), 5(3)^2, 5(3)^3$$

Or, 5, 15, 45, 135

### Formulae for a Geometric Progression

If the first term of a geometric progression is given as the letter  $a$  and the common ratio given as  $r$ , then:

1. the *nth* -term can be determined by:  $a_n = ar^{n-1}$
2. *Sum* of the first  $n$  terms can be obtained by:

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$$

If the absolute value of  $r$ ,  $|r| < 1$ , the sum to infinity is:

$$S_\infty = a / (1 - r)$$

### Examples

1. Can you find the 7<sup>th</sup> term and the sum of the first 10 terms of the geometric progression: 5, 15, 45, 135 ...

### Solution

*First*, you should identify this series a geometric progression with the first term,  $a = 5$ , and common ratio,  $r = 3$ . Then using the formula for the *nth*-term of a geometric progression, you get:

$$a_n = ar^{n-1}, \text{ where } n = 7$$

$$a_7 = 5(3)^{7-1} = 5(3)^6 = 3,645$$

Using the formula for the sum of a geometric progression, you get:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\begin{aligned}
 \text{Sum of the first 10 terms} = S_{10} &= \frac{5[(3)^{10} - 1]}{3 - 1} \\
 &= \frac{5[59049 - 1]}{2} \\
 &= \frac{295,240}{2} \\
 &= 147,620
 \end{aligned}$$

2. Find the *sum to infinity* of the geometric progression:

16, 4, 1, 0.25 ... etc

### Solution

Notice that in this progression,  $a = 16$  and  $r = 4/16 = 1/4 = \dots$   
 $= 0.25 < 1$

Recall that the sum to infinity is formulated as:

$S_{\infty} = a / (1 - r)$ , since  $r < 1$  in this example.

Thus  $S_{\infty} = 16 / (1 - 0.25) = 16 / 0.75 = 21.33$ .

### SELF ASSESSMENT EXERCISE 1

1. Discuss briefly the basic difference between Arithmetic and Geometric Progressions
2. If the first term and common ratio of a geometric progression are denoted by  $a$  and  $r$ , respectively, what is the expression used in calculating the sum of the first  $n$  terms?

### 4.0 CONCLUSION

This unit has presented the basic principles of progressions, with special emphasis on arithmetic and geometric progressions. The major conclusions you can draw from the discussions are that:

1. the most important applications of progressions in business decisions are in the areas of determination of the numeric values of any term and sum of a given mathematical series.
2. the basic difference between an arithmetic progression and a geometric progression is in their constant factors: the common

difference for an arithmetic progression; and, the common ratio in a geometric progression.

3. identification of the first term and the common factors will enable you know whether a given numeric series is an arithmetic series or a geometric series.

## 5.0 SUMMARY

The major points you may have learned from this unit can be summarised as follows:

1. An arithmetic progression is of the form:  $a, a + d, a + 2d, A + 3d$ . where  $d$  represents the common difference, and  $a$  represents the first term.
2. The  $n$ -th of an arithmetic progression (AP) is given by:  
 $a_n = a + (n - 1)d$
3. The sum of the first  $n$  terms of an AP is given by:  
 $S_n = n/2[2a + (n - 1)d]$
4. A geometric progression is of the form:  $a, ar, ar^2, ar^3 \dots$  where  $a$  is the first term and  $r$  represents the common ratio.
5. The  $n$ -th term of a geometric progression is obtained by:  
 $a_n = ar^{n-1}$
6. The sum of the first  $n$ -terms of a geometric progression is obtained by:  
$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$$
7. Sum of a geometric progression to infinity can be obtained by:  
 $S_\infty = a / (1 - r)$

## 6.0 TUTOR-MARKED ASSIGNMENT

- (1) Find the sum of the first 15 terms of the progression:  
48, 40, 32, 24 ...
- (2) Find the 6<sup>th</sup> term and the sum of the first 8 terms of the progression: 1, 3, 9, 27 ...
- (3) Find the sum to infinity of the progression:  
14,  $14(0.8)$ ,  $14(0.8)^2$ ,  $14(0.8)^3 \dots$

## **7.0 REFERENCES/FURTHER READING**

A. Francis (1998). *Business Mathematic and Statistics, 5<sup>th</sup> edition* (Great Britain: Ashford Colour Press).

## UNIT 2 INTEREST RATE AND DEPRECIATION

### CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
  - 3.1 Simple Interest, Compound Interest, and Depreciation
    - 3.1.1 Simple Interest
    - 3.1.2 Compound Interest
  - 3.2 Formulae for Amount Accrued: Simple and Compound Interest
    - 3.2.1 Accrued Amount for Simple Interest
    - 3.2.2 Accrued Amount for Compound Interest
  - 3.3 Nominal and Effective Interest Rates
  - 3.4 Depreciation
    - 3.4.1 The Straight Line Depreciation
    - 3.4.2 Reducing Balance Depreciation
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

### 1.0 INTRODUCTION

The terms interest rate and depreciation are fundamental concepts in business and economics. Interest is earned on money you invested and paid on money you borrowed. To discuss interest rates and depreciation in an understandable manner, you need to understand the principles behind progressions, and the differences between simple and compound interest. This unit will put these issues together for you to understand.

### 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify the differences between simple and compound interest and how they can be calculated
- explain nominal and effective interest rates
- recognise the basic concept of depreciation and how it can be calculated.

### 3.0 MAIN CONTENT

#### 3.1 Simple Interest, Compound Interest, and Depreciation

If a given amount of money is invested over a number of years, the interest earned can be dealt with in two ways:

##### 3.1.1 Simple Interest

In simple interest calculations, interests earned are not added back to the principal amount invested.

For **example**, suppose N1000 is invested at 15 percent simple interest per year. You can use the following table to show the state of the investment on a yearly basis:

**Table 17.1: Annual Interest Earnings on Investment**

Year	Principal Amount (P)	Interest Earned 15% of P	Cumulative Amount Accrued (P + interest Earned)
1	N1000	N150	$N(1000 + 150) = N1,150$
2	N1000	N150	$N(1,150 + 150) = N1,300$
3	N1000	N150	$N(1,300 + 150) = N1,450$
---			
---			
etc			

##### 3.1.2 Compound Interest

In compound interest calculations, you add interest earned back to the previous amount accrued.

Example, suppose that the N1000 is invested at 15 percent compound interest, the state of this investment can be shown annually according to table 17.2 below:

**Table 17.2: Investment Compounded at 15% annually**

Year	Principal Amount (P)	Interest Earned 15% of P	Cumulative Amount Accrued
1	N1000	15% of 1000 = N150	N1,150
2	N1,150	15% of 1150 =	N1,172.50
3	N1,172.50	N172.50	N1,348.38
...		15% of 1172.5 =	
		N175.88	

### 3.2 Formulae for Amount Accrued: Simple and Compound Interest

#### 3.2.1 Accrued Amount for Simple Interest

$$A_n = P (1 + nr)$$

Where:

$A_n$  = the accrued amount at end of the n-th year

P = Principal amount

r = annual rate of interest

n = number of years

Note however, that simple interest is of no practical value in modern business situations, since in practice, interest earnings are always compounded.

#### 3.2.2 Accrued Amount for Compound Interest

If a principal amount, P is invested at a compound interest rate of r percent annually over n years then:

$$A_1 = P + rP = \text{total amount accrued at the end of the first year}$$

$$= P (1 + r), \text{ by factorisation}$$

$$A_2 = P (1 + r) + P (1 + r) r = \text{accrued amount at year 1 times } r$$

$$= P (1 + r)(1 + r) \text{ by factorisation}$$

$$= P (1 + r)^2$$

Similarly,  $A_3 = P (1 + r)^3$

$A_4 = P (1 + r)^4$ , and so on.

For **example**, Assume N50, 000 ( $P = N50, 000$ ) is invested by you in the form of Certificate of Deposit (CD) at 10 percent per year ( $r = .10$ ), then:

Amount accrued after **1 year** would be:

$$\begin{aligned} A_1 &= P(1 + r) \\ &= 50,000(1 + 0.10) = 50,000(1.10) = N55, 000 \end{aligned}$$

Amount accrued after **two** years:

$$A_2 = P (1 + r)^2 = 50,000(1.10)^2 = N60,500$$

Amount accrued after **three** years:

$$A_3 = P (1 + r)^3 = 50,000(1.10)^3 = N66,550,$$

and so on.

In general terms, if the principal amount,  $P$ , is invested at  $r$  percent compounded annually. Then the amount accrued over  $n$  number of years can be calculated by the formula:

$$A_n = P (1 + r)^n$$

where  $A_n$  = the accrued amount at the end of  $n$ -th year

$P$  = the principal amount

$r$  = the annual rate of interest

$n$  = number of years.

Note the following points:

1. The above accrued amount formula can be transposed to make the principal,  $P$ , the subject by solving for  $P$  as follows:

$$P = A / (1 + r)^n$$

It follows that, given an interest rate,  $r$ , the accrued amount,  $A$ , and the time period involved,  $n$ , the principal amount can be obtained if you know the accrued amount.

For **example**, using our previous example, you may want to calculate the principal amount, given that the annual rate of interest is 10 percent ( $r = 0.10$ ), the accrued amount,  $A = N66, 550$ , and the number of years of interest is  $n = 3$  years. You can calculate the principal amount as follows:

$$P = A / (1 + r)^n$$



$$\begin{aligned}
 &= 66,500 / (1 + 0.10)^3 \\
 &= 66,500 \\
 &\quad 1.33 \\
 &= \text{N}50,000
 \end{aligned}$$

2. The standard time period for the calculation of interest earning is usually 1 year, so that the value of  $n$  in the formulae is assumed to be the number of years, unless otherwise stated.

### 3.3 Nominal and Effective Interest Rates

It is a normal practice in business to express interest rates as figures per annum, even though the interest may be compounded over time periods of less than 1 year. The given annual rate in this case is referred to as **nominal rate**.

Compounding may be six-monthly, quarterly, monthly, or daily. The actual annual rate of interest, referred to as **effective rate** or **actual percentage rate (APR)**, is **always greater** than the **nominal rate**.

The standard method of determining the **APR** is to make the *effective time* period *equal* to the *compounding* period and actually compound over a 1 year period.

To reflect this proposition, the following formula has been put together for your convenience:

Given the nominal annual rate of interest, the effective rate or actual percentage rate, APR, can be calculated by:

$$\text{APR} = (1 + r/n)^n - 1$$

Where  $r$  is the given annual rate of interest and,  $n$  is the number of equal compounding periods in 1 year.

#### Examples

1. A 10 percent nominal rate compounded quarterly will give you an APR of  $(1 + 0.10/4)^4 - 1 = (1.025)^4 - 1 = 1.1038 - 1 = 10.38\%$
2. A 10 percent nominal rate compounded monthly will give you an APR of  $(1 + 0.10/12)^{12} - 1 = 1.008^{12} - 1 = 1.1003 - 1 = 10.03\%$ .

### Other Examples

Consider the following business situation: A company wanted to spend N300, 000 on a new plant in two years from this point. The current nominal investment rates are at 10 percent per annum:

1. What single sum should be invested now if compounding is six-monthly?
2. What is your estimate of the actual percentage rate, APR?

### Solutions

1. Since the compounding period is six-monthly, the investment which you can denote as P will accrue to a value of N300,000 after two years (four six-monthly) periods. You can observe that the interest rate for each six-month period is  $(10/2) = 5$  percent.

Use the compounding (accrued amount) formula, you get:

$300,000 = P (1 + 0.05)^4$ , and solving for P you get:

$$\begin{aligned} P &= \frac{300,000}{(1+0.05)^4} \\ &= 300,000 / (1.05)^4 \\ &= 300000 / 1.216 = \text{N}246, 710.53. \end{aligned}$$

This implies that the amount to be invested is N246, 710.53 approximately.

2. When you apply the actual percentage rate (APR) formula, you will get:

$$\begin{aligned} \text{APR} &= (1 + r/n)^n - 1 \\ &= (1 + 0.10/2)^2 - 1 \\ &= (1 + 0.05)^2 - 1 \\ &= (1.05)^2 - 1 = 0.1025 \text{ or } 10.25\% \end{aligned}$$

### 3.4 Depreciation

You can refer to '*depreciation*' as an allowance made in estimate, valuation or accounting balance sheets, usually for wear and tear. Two of the various methods used in calculating depreciation include:

1. **Straight Line** (or equal installment) depreciation; and,
2. **Reducing Balance** depreciation.

You can regard these two methods as the converse of the interest rate calculation techniques we discussed earlier. This means that, instead of adding value to an original principal amount, P, as in the calculation of interest, some values are taken out in order to reduce the original amount. Straight line depreciation is the converse of simple interest. In straight line depreciation, amounts are subtracted, rather than added. In reducing balance depreciation method, you get the converse of compound interest.

#### 3.4.1 The Straight Line Depreciation

As you noticed earlier, the straight line depreciation method requires that you subtract the same amount from the original book value of investment each year. For instance, if the value of a given machine is to depreciate from N250,000 to N50,000 over a period of five years, then the yearly depreciation will be:

$$\begin{aligned} & \frac{N(250,000 - 50,000)}{5} \\ &= \frac{N200,000}{5} \\ &= N40,000 \end{aligned}$$

#### 3.4.2 Reducing Balance Depreciation

If you need to depreciate some values, say P, by r percent, you need to calculate  $P(1 - r)$ . You will notice that the multiplier is now  $1 - r$ , rather than  $1 + r$  as was with the case of compound interest calculations where you were calculating the value  $P(1 + r)$ . If you carry out the depreciation successively in n-times the accrued depreciation value will be  $P(1 - r)^n$ .

For example, if N250, 000 is depreciated by you by 10 percent ( $r = 0.10$ ), over four time periods ( $n = 4$ ), your final depreciated value will be:

$$\begin{aligned} & N \{250,000(1 - 0.10)^4\} \\ & = N \{250,000(0.90)^4\} \\ & = N164, 025. \end{aligned}$$

The formula for reducing balance depreciation can be written as follows:

$$D = P (1 - r)^n$$

Where  $D$  = depreciated value at the end of the  $n$ -th time period

$P$  = the original book value

$r$  = depreciation rate

$n$  = number of time periods, normally in years.

### SELF ASSESSMENT EXERCISE 1

1. If the sum of N560, 000 is invested at 16 percent simple interest. How much would have accrued after 8 years?
2. Find the amount of:
  - (a) N150, 000 compounded at 15 percent over 4 years
  - (b) N200, 000 compounded at 12.5 percent over 7 years.

## 4.0 CONCLUSION

This unit has revealed that interest rate and depreciation are fundamental concepts in business and economics. Interest is earned on money you invested and paid on money you borrowed. To discuss interest rates and depreciation in an understandable manner, you need to understand the principles behind progressions, and the differences between simple and compound interest. This and the immediate past unit have put these issues together for you to understand.

The unit also points out that the standard time period for the calculation of interest earning is usually 1 year, so that the value of  $n$  in the interest rate formulae is assumed to be the number of years, unless otherwise stated.

It is a normal practice in business to express interest rates as figures per annum, even though the interest may be compounded over time periods of less than 1 year. The given annual rate in this case is referred to as **nominal rate**.

Compounding may be six-monthly, quarterly, monthly, or daily, and, the actual annual rate of interest, referred to as **effective rate** or **actual percentage rate (APR)**, is **always greater** than the **nominal rate**.

The standard method of determining the **APR** is to make the *effective time* period *equal* to the *compounding* period and actually compound over a 1 year period.

## 5.0 SUMMARY

The major issues presented in this unit can be summarised as follows:

1. In simple interest calculations, interest earned are not added back to a principal amount, while in compound interest calculation, interest earned is added back to the previously accrued amount
2. The actual percentage rate of interest (APR) is always greater than the nominal rate and is formulated by:

$$\text{APR} = (1 + r/n)^n - 1.$$

The term “depreciation” refers to an amount made in estimate, valuation or accounting balance sheet. There are two major techniques employed in the calculation of depreciation, including:

1. The straight line depreciation technique, requiring that you subtract some amount from the original book value of a given investment each year.
2. Reducing balance depreciation, which is a direct opposite of the computational process for compounding interest. The formula for this type of depreciation is worth noting again:

$$D = P (1 - r)^n = \text{the depreciated value at end of the } n\text{-th time period.}$$

## 6.0 TUTOR-MARKED ASSIGNMENT

A business firm invests N400, 000 per year (at the end of every year) at 15 percent compounded to meet a fixed obligation at a particular time period. If the obligation is N800, 000 to be paid in exactly three years from now, what single sum to be invested now needs to be added in order to meet the obligation.

## 7.0 REFERENCES/FURTHER READING

- A. Francis (1998). *Business Mathematic and Statistics, 5<sup>th</sup> edition* (Great Britain: Ashford Colour Press).

## **UNIT 3     PRESENT VALUE AND INVESTMENT APPRAISALS**

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### **1.0 INTRODUCTION**

The technique of present value and how it can be applied to future cash flows in order to find their worth in today's money value are becoming essential for any productive real investment outlays. The concept 'Present Value' enables businesses calculate discounted cash flows for investments. Another important tool needed for investment decision is that of investment appraisal method. In this unit, these issues will be thoroughly addressed, with the aim of acquainting you with the basic tools of investment appraisals.

### **2.0 OBJECTIVES**

At the end of this unit, you should be able to:

- describe the present value concept
- discuss investment appraisal techniques
- calculate present values
- define capital investments and the discounted cash flow
- identify concept of internal rate of return (IRR)
- explain profitable investment decisions.

### 3.0 MAIN CONTENT

#### 3.1 Calculation of the Present Value

You can imagine that some money is invested at 10 percent per year. Then, you can invest N10, 000 and earn N1000 in one year time period, and your investment will be worth N11, 000. Or put it differently, the value of N11, 000 in one year time is exactly the same as N10, 000 now. This exemplifies the concept of the present value of a future sum.

To state this precisely, if the current investment rate is 10 percent, then the present value of N11, 000 in one year's time would be:

$$N (11,000/1.1) = N10, 000$$

Similarly, the present value of N20, 000 in two years time would be:

$$N (20,000/(1.1)^2) = N16,528.93, \text{ and so on.}$$

The **investment rate**, which is 10 percent in this illustration, is referred to as the **discount rate**. The technique for calculation of present value can be generalised using the following formula:

The present value of an amount NA, payable in n year's time, subject to a discount rate of r percent can be obtained by:

$$PV = A / (1 + r)^n,$$

Where PV = present value

A = amount payable in n year's time

r = the discount rate

n = the number of time periods in years.

The term in the above formula,  $A / (1 + r)^n$  is known as the **present value factor** or the **discount factor**.

#### 3.2 Present Value of an Interest-Bearing Debt

The amount of money borrowed by you will always attract interest at an appropriate borrowing rate. Assuming there are no intermediate payments made by you, at the end of the period involved, the amount of debt plus the interest must be repaid by you as a lump sum. This requires the calculation of present value of such amount.

Thus, given that the amount borrowed is say, P at r percent interest compounded for n years, this debt will amount to  $P(1 + r)^n$ . If the investment or discount rate is j percent, the debt amount would have to

be discounted at  $j$  percent back over the  $n$  years. Thus the present value of the interest-bearing debt is given by the formula:

$$PV = \frac{P(1+r)^n}{(1+j)^n}$$

Where  $r$  = the borrowing rate  
 $j$  = the discount rate  
 $P$  = the original amount borrowed  
 $n$  = the number of time periods.

Note that the borrowing rate will always be greater than the investment rate, that is,  $r > j$ , so that the ratio:

$$\frac{P(1+r)^n}{(1+j)^n}$$

will always be greater than 1. This is to say that the present value of the debt (PV) will always be greater than  $P$  (the original amount borrowed).

### Example

Can you find the present value of a debt valued at N250, 000 taken out over 3 years, with no intermediate payments, given that the borrowing rate is 15 percent per year and the discount rate or the worth of money is 10 percent.

### Solution

The maturity value of the debt will be calculated as:

$$P(1+r)^3 = 250,000(1+0.15)^3 = N[250,000(1.15)^3] = N3802,218.75$$

Therefore, the present value becomes:

$$\begin{aligned} PV &= \frac{3,802,218.75}{(1.10)^3} \\ &= N2,856,663.22 \end{aligned}$$

This means that the original debt of N250, 000 will cost (in today's monetary terms) N2, 856,663.22 to repay.



### 3.3 Capital Investment Appraisals

In its simplest form, capital investment is a project consisting of:

- (i) an initial outlay
- (ii) a set of estimated cash inflows and outflows over the life of the project
- (iii) optionally, a resettlement figure which might be as a result of plant resale or shares, or cash settlement for clearing any liabilities in the cause of project execution.

There are many ways you can appraise and compare capital investments, but we look at two ways or techniques, both using the concept of discounting, including:

1. Discounted Cash Flow
2. Internal Rate of Return (IRR).

#### 3.3.1 Discounted Cash Flow

The discounted cash flow technique involves calculating the sum of the present values of all cash flows associated with an investment project. This sum is referred to as the *Net Present Value (NPV)* of the project. The cash flows are tabulated net per year. The standard presentation format is illustrated by the example that follows.

#### Example

Assume that a business project is being considered, with initial costs of N12,000 and corresponding revenues or inflows over the following 4 years of N8,000, N12,000, N10,000, and N6,500, respectively. If the project costs (outflows) over the 4 years are estimated at N8,500, N3,000, N1,500, and N1,500, respectively, and the discount rate is 10 percent, evaluate the project's Net Present Value (NPV).

#### Solution

Tables 18.1 and 18.2 below present the project's cash flows and the discounted cash flows respectively. Table 18.2 yields the net present value of the project.

**Table 18.1: The Cash Flows**

Year	Cash Inflow (a)	Cash Outflow (b)	Net Cash Flow (a) – (b)
0	-	12000	(12000)
1	8000	8500	(500)
2	12000	3000	9000
3	10000	1500	8500
4	6500	1500	5000

**Table 18.2: The Discounted Cash Flow**

Year	Net Cash Flow	Discount Factor At 10 %	Present Value
0	(12000)	$1/(1.10)^0 = 1.0000$	(12000)
1	(500)	$1/(1.10)^1 = 0.9091$	(454.55)
2	9000	$1/(1.10)^2 = 0.8265$	7438.50
3	8500	$1/(1.10)^3 = 0.7513$	6386.05
4	5000	$1/(1.10)^4 = 0.6830$	3415.00
			<b>NPV = 3415 + 6386.05 + 7438.50 – 454.55 – 12000 = N4785.00</b>

In general,

If  $NPV > 0$ , the project is profitable and should be undertaken

If  $NPV = 0$ , the project breaks even, and

If  $NPV < 0$ , the project will make a loss. It is not worthwhile and  
It should not be undertaken.

Alternatively,

If  $NPV > 0$ , the project earns more than the discount rate and it is  
worthwhile

If  $NPV = 0$ , the project earns the same as the discount rate and,

If  $NPV < 0$ , the project earns less than the discount rate and it is not  
worthwhile.

### 3.3.2 The Internal Rate of Return (IRR)

The internal rate of return (IRR) is another technique used for investment appraisals. The IRR of a given investment project can be

defined as the value of the discount rate that gives a zero Net Present Value (NPV), (that is,  $NPV = 0$ ). The internal rate of return can also be viewed as the rate earned by a given project.

Presently, there does not exist a precise formula for calculating the internal rate of return for a project. Nevertheless, IRR can be estimated using a linear interpolation formula as presented below:

$$IRR = \frac{N_1 R_2 - N_2 R_1}{N_1 - N_2}$$

Where the discount rate,  $R_1$  gives NPV of  $N_1$ , and the discount rate  $R_2$  gives NPV of  $N_2$

### Example

Let the discount rate,  $R_1 = 15\%$  yield an NPV of N140, 000 (that is,  $R_1 = 0.15$  and  $N_1 = 140,000$ ); and, the discount rate of  $R_2 = 17\%$  yield an NPV of -N70, 000 (  $R_2 = 0.17$  and  $N_2 = -70,000$ ). It follows that:

$$\begin{aligned} IRR &= \frac{(140,000)(0.17) - (-70,000)(0.15)}{140,000 - (-70,000)} \\ &= \frac{23,800 + 10,500}{140,000 + 70,000} \\ &= \frac{34,300}{210,000} \\ &= 0.1633 \text{ or } 16.33 \text{ percent.} \end{aligned}$$

Therefore, the estimate of the internal rate of return (IRR) is approximately 16.33 percent. This is the discount rate that yields  $NPV = 0$ .

In the following example, you will see the use of the above formula from a discount cash flow table.

### Example

Using table 18.3 below, we wish to calculate the NPV corresponding to each of two discount rates, 6 percent and 9 percent, summing up to 15 percent. We also estimate the internal rate of return of the project.

**Solution**

When calculating the NPVs of a project using two separate discount rates, you should combine both sets of calculations into a single table. This dual discounted cash flow table for the project is presented in table 18.3 below.

**Table 18.3: The Dual Discounted Cash Flow**

<b>Year</b>	<b>Net Flow</b>	<b>Discount factor at 6%</b>	<b>Present Value</b>	<b>Discount factor at 9%</b>	<b>Present Value</b>
0	(60,000)	1.0000	(60,000)	1.0000	(60,000)
1	(10,000)	0.9434	(9,434)	0.9174	(9,174)
2	15,000	0.8900	13,350	0.8417	12,625.50
3	20,000	0.8396	16,792	0.7722	15,444
4	20,000	0.7921	15,842	0.7084	14,168
5	20,000	0.7473	14,946	0.6499	12,998
6	20,000	0.7050	14,100	0.5963	11,926
		<b>NPV</b>	<b>5,596</b>		<b>(2,012.50)</b>

Using the formula, we estimate the IRR as follows:

$$IRR = \frac{N_1 R_2 - N_2 R_1}{N_1 - N_2}$$

Where  $N_1 = 5596$ ;  $N_2 = -2,012.50$ ;

$$R_1 = 0.06; R_2 = 0.09$$

$$IRR = \frac{(5596)(0.09) - (-2012.50)(0.06)}{5596 - (-2012.50)}$$

$$= \frac{503.64 + 120.75}{5596 + 2012.50}$$

$$= \frac{524.39}{7608.50}$$

$$= 0.082065$$

= 0.0821 or 8.21 percent.

It follows that the internal rate of return (IRR) is about 8.21 percent which is below the required 15 percent. The **project should therefore be rejected.**

### **3.4 Comparisons of Different Projects and Different Appraisal Techniques**

This section examines how you can compare two or more different investment or project alternative, as well as how you can compare two or more different appraisal techniques. These comparisons will help in the choice of the best investment alternative and the best appraisal technique.

#### **3.4.1 Comparison of Projects**

It appears there is no standard formula for deciding which among a number of projects the best choice is. At times a choice will be obvious, and at other times, it will not be clear. This depends on a particular business circumstance with regard to the following factors.

1. Whether capital is available or it needs to be borrowed,
2. The ability to pay rates of interest,
3. General liquidity,
4. Uncertainty of estimated flow.

Given that these factors are not significantly different for two or more similar projects, both NPV and IRR can be used for the comparisons. This can be done according to the following discussions.

#### **3.4.2 Comparison of Appraisal Techniques**

##### **1. The Net Present Value (NPV)**

You will normally choose the project that has the largest NPV as your most profitable project. It appears the NPV technique is the most suitable to projects with similar pattern of cash flows over the same length of time.

The **advantage** of the use of NPV as project appraisal technique is that it is practical and relevant as it discounts net flows. In this technique, further adjustments can be made to take account of factors such as inflation and taxation. In addition, this technique gives results in money terms.

The **disadvantage** however, is that it relies on the choice of a discount factor. In some cases, where negative cash flows occur during the life of a project, changing the discount rate might change a previous choice of project.

## 2. The Internal Rate of Return (IRR)

Using the IRR technique, the project with the highest internal rate of return will be chosen. The major **advantage** of this method of investment appraisal is that it does not depend on any external rate of interest, whereas the NPV depends on the choice of a discount rate. The **disadvantage** is that the IRR returns a relative (percentage) value and thus does not differentiate between the scales of projects. This is to say that one project could involve cash flows in units of N100, 000, the other in units of N100, which is of significance for some project comparisons.

### Example

The followings are two capital projects involving the purchase, use and final disposal of two machines, machine A and machine B:

Machine Type	Initial Cost	Net Cash Flows			
		Year 1	Year 2	Year 3	Year 4
Machine A	50,000	25,500	24,500	17,000	14,000
Machine B	45,000	12,500	15,500	21,000	38,000

You are required to choose between projects A and B using in turn each of the following appraisal techniques:

- (a) Net Present Value (NPV), using a cost of capital of 22% (and 28%).
- (b) Internal Rate of Return (IRR), to be estimated using the results from (a).

## Solutions

The solutions are provided with the aid of tables 18.4 that follows.

**Table 18.4: Calculation for the Required Costs of Capital for Machines A and B**

Cash Flows for Machine A					
Year	Net Flow	Discount factor at 22%	Present Value	Discount factor at 28%	Present Value
0	(50,000)	1.0000	(50,000)	1.0000	(50,000)
1	25,500	0.8197	20,902.25	0.7813	19,923.15
2	24,500	0.6719	16,461.54	0.6104	14,954.79
3	17,000	0.5507	9,361.90	0.4768	8,105.59
4	14,000	0.4514	6,319.60	0.3725	5,215.00
		<b>NPV</b>	<b>3,045.40</b>		<b>(1,801.45)</b>

Cash Flows for Machine B					
Year	Net Flow	Discount factor at 22%	Present Value	Discount factor at 28%	Present Value
0	(45,000)	1.0000	(45,000)	1.0000	(45,000)
1	12,500	0.8197	10,246.25	0.7813	9,766.25
2	15,500	0.6719	10,414.45	0.6104	9,461.19
3	21,000	0.5507	11,564.70	0.4768	10,012.80
4	38,000	0.4514	17,153.19	0.3725	14,155.00
		<b>NPV</b>	<b>4,378.60</b>		<b>(1,604.75)</b>

Observe that machine B has the highest NPV at 22%. You will then choose machine B as the best alternative. Note also that machine B has a higher NPV at 28% than that of machine A.

In calculating the internal rate of return (IRR), you observe as follows:

For **machine A**, with  $R_1 = 0.22$ ;  $N_1 = 3045.40$ ;  $R_2 = 0.28$  and  $N_2 = -1801.45$ , the IRR is given by:

$$\begin{aligned} \text{IRR} &= \frac{(3045.40)(0.28) - (-1801.45)(0.22)}{3045.40 - (-1801.45)} \\ &= \frac{852.712 + 396.319}{3045.40 + 1801.45} \end{aligned}$$

$$= \frac{1249.031}{4846.85}$$

$$= 0.2577 \text{ or } 25.8 \text{ percent}$$

For machine B, with  $R_1 = 0.22$ ;  $N_1 = 4378.60$ ;  $R_2 = 0.28$ ;  $N_2 = -1604.75$ , the IRR is given by:

$$\text{IRR} = \frac{(4378.60)(0.28) - (-1604.75)(0.22)}{4378.60 - (-1604.75)}$$

$$= \frac{1226.01 + 3530.5}{4376.6 + 1604.70}$$

$$= \frac{1579.06}{5981.30}$$

$$= 0.264 \text{ or } 26.4 \text{ percent.}$$

It follows that machine B has a higher rate of return (IRR = 26.4%) than machine A, with IRR = 25.8%. Overall, machine B is clearly the best choice.

### SELF ASSESSMENT EXERCISE 1

1. What do you understand by Net Present Value (NPV) and Internal Rate of Return (IRR)? How are they used?
2. Why do you think it is useful to know the present value of an interest-bearing debt?
3. What are the basic methods available for appraising and comparing capital investments?

### 4.0 CONCLUSION

This unit has brought to bear the technique of present value and how it can be applied to future cash flows in order to find their worth in today's money value. We noted that this is highly essential for any productive real investment outlays. The concept 'Present Value' enables businesses calculate discounted cash flows for investments. Another important tool needed for investment decision is that of investment appraisal method. There are two basic investment appraisal techniques including, the net present value (NPV) and the internal rate of return (IRR). We discovered that the best choice of investment alternative is the alternative with either the highest NPV or the highest IRR or both as the case maybe.



## 5.0 SUMMARY

The information you obtained from unit 16 can be summarised as follows:

1. The present value (PV) technique enables you obtain a future cash flow which will be equivalent to today's monetary value. The present value is obtained by multiplying the cash flow by a discount factor.
2. The present value of an interest bearing debt can be obtained by:
  - (a) compounding the debt over an appropriate time period, using an appropriate borrowing rate;
  - (b) discounting back to today, using an appropriate investment rate.
3. The Net Present Value (NPV) of a project can be obtained by discounting all cash flows and adding them together, and:  
  
If  $NPV > 0$ , the project will be profitable  
If  $NPV = 0$ , the project breaks even  
If  $NPV < 0$ , the project is not profitable and not worthwhile.
4. The internal rate of return (IRR) of a project is that discount rate for which the  $NPV = 0$ . It is the rate of interest earned by a given project.
5. Investment appraisals using discounting techniques involves considering the worth of one or more capital projects, using either net present value (NPV) or internal rate of return (IRR).

## 6.0 TUTOR-MARKED ASSIGNMENT

A business outfit is considering borrowing N100, 000 to invest in a project, which is expected to yield N350, 000 at the end of each of the next 5 years. If the cost of capital is 25 percent:

- (a) draw up a discount cash flow table and calculate the NPVs of the project.
- (b) interpret your calculated value of NPV in light of the given situation.

## 7.0 REFERENCES/FURTHER READING

A. Francis (1998). *Business Mathematic and Statistics, 5<sup>th</sup> edition* (Great Britain: Ashford Colour Press).